

Integrals and crossed products over weak Hopf algebras

J.N. Alonso Álvarez¹, J.M. Fernández Vilaboa², R. González Rodríguez³

¹ Departamento de Matemáticas, Universidad de Vigo, Campus Universitario Lagoas-Marcosende, E-36280 Vigo, Spain (e-mail: jnalonso@uvigo.es)

² Departamento de Álgebra, Universidad de Santiago de Compostela. E-15771 Santiago de Compostela, Spain (e-mail: josemanuel.fernandez@usc.es)

³ Departamento de Matemática Aplicada II, Universidad de Vigo, Campus Universitario Lagoas-Marcosende, E-36310 Vigo, Spain (e-mail: rgon@dma.uvigo.es)

Abstract In this paper we present the general theory of cleft extensions for a cocommutative weak Hopf algebra H . For a weak left H -module algebra we obtain a bijective correspondence between the isomorphism classes of H -cleft extensions $A_H \hookrightarrow A$, where A_H is the subalgebra of coinvariants, and the equivalence classes of crossed systems for H over A_H . Finally, we establish a bijection between the set of equivalence classes of crossed systems with a fixed weak H -module algebra structure and the second cohomology group $H^2_{\varphi_{Z(A_H)}}(H, Z(A_H))$, where $Z(A_H)$ is the center of A_H .

Keywords. Monoidal category, weak Hopf algebra, cleft extension, weak crossed product, Sweedler cohomology for weak Hopf algebras.

MSC 2010: 18D10, 16T05.

1. INTRODUCTION

Weak Hopf algebras (or quantum groupoids in the terminology of Nikshych and Vainerman [24]) have been introduced by Böhm, Nill and Szlachányi [9] as a new generalization of Hopf algebras and groupoid algebras. Roughly speaking, a weak Hopf algebra H in a symmetric monoidal category is an object which has both algebra and coalgebra structures with some relations between them and that possesses an antipode λ_H which does not necessarily satisfy the usual convolution equalities with the identity morphism. The main differences with other Hopf algebraic constructions, such as quasi-Hopf algebras and rational Hopf algebras, are the following: weak Hopf algebras are coassociative but the coproduct is not required to preserve the unity morphism or, equivalently, the counit is not an algebra morphism. Some motivations to study weak Hopf algebras come from their connection with the theory of algebra extensions, the important applications in the study of dynamical twists of Hopf algebras and their link with quantum field theories and operator algebras (see [24]). It is well-known that groupoid algebras of finite groupoids provides examples of weak Hopf algebras. If G is a finite groupoid (a category with a finite number of objects such that each morphism is invertible) then the groupoid algebra over a commutative ring R is an example of cocommutative weak Hopf algebra. This weak Hopf algebra, denoted by RG , is generated by the morphisms g in G and the product of two morphisms is defined by the composition if it exists and 0 in other case. The coalgebra structure is defined by the coproduct $\delta_{RG}(g) = g \otimes g$, and the counit $\varepsilon_{RG}(g) = 1$ and the antipode by $\lambda_{RG}(g) = g^{-1}$. There are more interesting examples of cocommutative weak Hopf algebras, for example recently Bulacu in [13, 14] proved that Cayley-Dickson and Clifford algebras are commutative and cocommutative weak Hopf algebras in some suitable symmetric monoidal categories of graded vector spaces.

Like in the Hopf algebra setting, it is possible to define a theory of crossed products for weak Hopf algebras. The key to extend the crossed product constructions of the Hopf world to the weak setting is the use of idempotent morphisms combined with the ideas giving in [11]. In [4, 19], the authors defined a product on $A \otimes V$, for an algebra A and an object V both living in a strict monoidal category \mathcal{C} where every idempotent splits. In order to obtain that product we must consider two morphisms $\psi_V^A : V \otimes A \rightarrow A \otimes V$ and $\sigma_V^A : V \otimes V \rightarrow A \otimes V$ that satisfy some twisted-like and cocycle-like conditions. Associated to these morphisms it is possible to define an idempotent morphism $\nabla_{A \otimes V} : A \otimes V \rightarrow A \otimes V$, that becomes the identity in the classical case. The image of this idempotent inherits the associative product from $A \otimes V$. In order to define a unit for $Im(\nabla_{A \otimes V})$, and hence to obtain an algebra structure, we require

the existence of a preunit $\nu : K \rightarrow A \otimes V$ and, under these conditions, it is possible to characterize weak crossed products with an unit as products on $A \otimes V$ that are morphisms of left A -modules with preunit. Finally, it is convenient to observe that, if the preunit is an unit, the idempotent becomes the identity and we recover the classical examples of the Hopf algebra setting. The theory presented in [4, 19] contains as a particular instance the one developed by Brzeziński in [11] as well as all the crossed product constructed in the weak setting, for example the ones defined in [15], [25] and [21]. Recently, G. Böhm showed in [10] that a monad in the weak version of the Lack and Street's 2-category of monads in a 2-category is identical to a crossed product system in the sense of [4] and also in [20] we can find that unified crossed products [1] and partial crossed products [23] are particular instances of weak crossed products. An interesting example of weak crossed product comes from the theory of weak cleft extensions associated to weak Hopf algebras. This notion was introduced in [2] and in [4] we show that it provides an example of weak crossed product satisfying the weak twisted and the cocycle conditions. These crossed products are deeply connected with Galois theory as we can see in the intrinsic characterization of weak clefthness in terms of weak Galois extensions with normal basis obtained in [22]. We want to point that, when we particularize this weak cleft theory to the Hopf algebra setting we obtain a more general notion than the usual of cleft extension (see Definition 7.2.1 of [22]) because in this case the uniqueness of the cleaving morphism is not guaranteed.

In the Hopf setting the theory of crossed products arise as a generalization of the classical smash products and by the results obtained by Doi and Takeuchi in [18] we know that every cleft extension $D \hookrightarrow A$ with cleaving morphism f such that $f(1_H) = 1_A$ induces a crossed product $D \#_\sigma H$ where $\sigma : H \otimes H \rightarrow D$ is a suitable convolution invertible morphism (a normal 2-cocycle). Conversely, in [8] we can find the reverse result, that is, if $D \#_\sigma H$ is a crossed product, the extension $D \hookrightarrow D \#_\sigma H$ is cleft. On the other hand, in [26] Sweedler introduced the cohomology of a cocommutative Hopf algebra H with coefficients in a commutative H -module algebra A . We will denote these cohomology groups as $H_{\varphi_A}(H^\bullet, A)$ where φ_A is a fixed action of H over A . In [26] we can find an interesting interpretation of the second cohomology group $H_{\varphi_A}^2(H, A)$ in terms of extensions: This group classifies the set of equivalence classes of cleft extensions, i.e., classes of equivalent crossed products determined by a 2-cocycle. This result was extended by Doi [17] proving that, in the non commutative case, there exists a bijection between the isomorphism classes of H -cleft extensions D of A and equivalence classes of crossed systems for H over A with a fixed action. If H is cocommutative the equivalence is described by $H_{\varphi_{\mathcal{Z}(A)}}^2(H, \mathcal{Z}(A))$ where $\mathcal{Z}(A)$ is the center of A .

The aim of this paper is to extend the preceding results to the cocommutative weak Hopf algebra setting completing the program initiated in [6]. To do it, in the second section, we introduce the notion of H -cleft extension for a weak Hopf algebra H and we prove that this kind of extensions are examples of weak cleft extensions as the ones introduced in [2] and satisfying that, when we particularize to the Hopf setting the classical notion used in the papers of Doi and Takeuchi is obtained. Also, we prove that, under cocommutative conditions, we can assume that the associated cleaving morphism is a total integral. In the third section, assuming that H is cocommutative, we prove that it is possible to identify the set of crossed systems in a weak setting as the set of weak crossed products induced by a weak left action and a convolution invertible twisted normal 2-cocycle and then, as a consequence, we obtain the main result of this section that assures the following: If (A, ρ_A) be a right H -comodule algebra, there exists a bijective correspondence between the equivalence classes of H -cleft extensions $A_H \hookrightarrow A$ and the equivalence classes of crossed systems for H over A_H where A_H denotes the subalgebra of coinvariants in the weak setting. Finally, in the fourth section we generalize the result obtained by Doi and Takeuchi about the characterization of equivalence classes of crossed systems using the second Sweedler cohomology group. To obtain this generalization we must use the cohomology theory of algebras over weak Hopf algebras that we have developed in [6]. The main result contained in [6] (see Theorem 3.11) asserts that if (A, φ_A) is a commutative left H -module algebra, there exists a bijection between the second cohomology group, denoted by $H_{\varphi_A}^2(H, A)$, and the equivalence classes of weak crossed products $A \otimes_\alpha H$ where $\alpha : H \otimes H \rightarrow A$ satisfy the 2-cocycle and the normal conditions. Then, by this bijection and using the results of the previous sections, we obtain the description of the bijection between the isomorphism classes of H -cleft extensions $A_H \hookrightarrow B$ and the equivalence classes of crossed systems for H over A_H in terms of $H_{\varphi_{\mathcal{Z}(A_H)}}^2(H, \mathcal{Z}(A_H))$.

2. INTEGRALS OVER WEAK HOPF ALGEBRAS

From now on \mathcal{C} denotes a strict symmetric category with tensor product denoted by \otimes and unit object K . With c we will denote the natural isomorphism of symmetry and we also assume that \mathcal{C} has equalizers. Then, under these conditions, every idempotent morphism $q : Y \rightarrow Y$ splits, i.e., there exist an object Z and morphisms $i : Z \rightarrow Y$ and $p : Y \rightarrow Z$ such that $q = i \circ p$ and $p \circ i = id_Z$. We denote the class of objects of \mathcal{C} by $|\mathcal{C}|$ and for each object $M \in |\mathcal{C}|$, the identity morphism by $id_M : M \rightarrow M$. For simplicity of notation, given objects M, N, P in \mathcal{C} and a morphism $f : M \rightarrow N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

An algebra in \mathcal{C} is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in \mathcal{C} and $\eta_A : K \rightarrow A$ (unit), $\mu_A : A \otimes A \rightarrow A$ (product) are morphisms in \mathcal{C} such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. We will say that an algebra A is commutative if $\mu_A \circ c_{A,A} = \mu_A$.

Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \rightarrow B$ is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$.

If A, B are algebras in \mathcal{C} , the object $A \otimes B$ is an algebra in \mathcal{C} where $\eta_{A \otimes B} = \eta_A \otimes \eta_B$ and

$$\mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B). \quad (1)$$

For an algebra A we define the center of A as a subalgebra $Z(A)$ of A with inclusion algebra morphism $i_{Z(A)} : Z(A) \rightarrow A$ satisfying

$$\mu_A \circ (A \otimes i_{Z(A)}) = \mu_A \circ c_{A,A} \circ (A \otimes i_{Z(A)})$$

and if $f : B \rightarrow A$ is a morphism such that $\mu_A \circ (A \otimes f) = \mu_A \circ c_{A,A} \circ (A \otimes f)$, there exists a unique morphism $f' : B \rightarrow Z(A)$ satisfying $i_{Z(A)} \circ f' = f$. As a consequence, we obtain that $Z(A)$ is a commutative algebra. For example, if \mathcal{C} is a closed category and α_A and β_A are the unit and the counit, respectively, of the \mathcal{C} -adjunction $A \otimes - \dashv [A, -] : \mathcal{C} \rightarrow \mathcal{C}$, the center of A can be obtained by the following equalizer diagram:

$$\begin{array}{ccccc} Z(A) & \xrightarrow{i_{Z(A)}} & A & \xrightleftharpoons[\theta_A]{\vartheta_A} & [A, A] \end{array}$$

where $\vartheta_A = [A, \mu_A] \circ \alpha_A(A)$ and $\theta_A = [A, \mu_A \circ c_{A,A}] \circ \alpha_A(A)$.

A coalgebra in \mathcal{C} is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in \mathcal{C} and $\varepsilon_D : D \rightarrow K$ (counit), $\delta_D : D \rightarrow D \otimes D$ (coproduct) are morphisms in \mathcal{C} such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. We will say that D is cocommutative if $c_{D,D} \circ \delta_D = \delta_D$ holds.

If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, $f : D \rightarrow E$ is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$ and $\varepsilon_E \circ f = \varepsilon_D$.

When D, E are coalgebras in \mathcal{C} , $D \otimes E$ is a coalgebra in \mathcal{C} where $\varepsilon_{D \otimes E} = \varepsilon_D \otimes \varepsilon_E$ and

$$\delta_{D \otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E). \quad (2)$$

If A is an algebra, B is a coalgebra and $\alpha : B \rightarrow A$, $\beta : B \rightarrow A$ are morphisms, we define the convolution product by

$$\alpha \wedge \beta = \mu_A \circ (\alpha \otimes \beta) \circ \delta_B.$$

Let A be an algebra. The pair (M, φ_M) is a left A -module if M is an object in \mathcal{C} and $\varphi_M : A \otimes M \rightarrow M$ is a morphism in \mathcal{C} satisfying $\varphi_M \circ (\eta_A \otimes M) = id_M$, $\varphi_M \circ (A \otimes \varphi_M) = \varphi_M \circ (\mu_A \otimes M)$. Given two right A -modules (M, φ_M) and (N, φ_N) , $f : M \rightarrow N$ is a morphism of right A -modules if $\varphi_N \circ (A \otimes f) = f \circ \varphi_M$.

Let C be a coalgebra. The pair (M, ρ_M) is a right C -comodule if M is an object in \mathcal{C} and $\rho_M : M \rightarrow M \otimes C$ is a morphism in \mathcal{C} satisfying $(M \otimes \varepsilon_C) \circ \rho_M = id_M$, $(M \otimes \rho_M) \circ \rho_M = (M \otimes \delta_C) \circ \rho_M$. Given two right C -comodules (M, ρ_M) and (N, ρ_N) , $f : M \rightarrow N$ is a morphism of right C -comodules if $(f \otimes C) \circ \rho_M = \rho_N \circ f$.

By weak Hopf algebras we understand the objects introduced in [9], as a generalization of ordinary Hopf algebras. Here we recall the definition of these objects in a monoidal symmetric setting.

Definition 2.1. A weak Hopf algebra H is an object in \mathcal{C} with an algebra structure (H, η_H, μ_H) and a coalgebra structure $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

- (a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H \otimes H}$,
- (a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$
 $= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$,
- (a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$
 $= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.
- (a4) There exists a morphism $\lambda_H : H \rightarrow H$ in \mathcal{C} (called the antipode of H) satisfying:
 - (a4-1) $id_H \wedge \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,
 - (a4-2) $\lambda_H \wedge id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,
 - (a4-3) $\lambda_H \wedge id_H \wedge \lambda_H = \lambda_H$.

Note that, in this definition, the conditions (a2), (a3) weaken the conditions of multiplicativity of the counit, and comultiplicativity of the unit that we can find in the Hopf algebra definition. On the other hand, axioms (a4-1), (a4-2) and (a4-3) weaken the properties of the antipode in a Hopf algebra. Therefore, a weak Hopf algebra is a Hopf algebra if and only if the morphism δ_H (comultiplication) is unit-preserving or if and only if the counit is a homomorphism of algebras.

2.2. If H is a weak Hopf algebra in \mathcal{C} , the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit η_H and the counit ε_H invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}; \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H; \quad (3)$$

$$\lambda_H \circ \eta_H = \eta_H; \quad \varepsilon_H \circ \lambda_H = \varepsilon_H. \quad (4)$$

If we define the morphisms Π_H^L (target), Π_H^R (source), $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$$\begin{aligned} \Pi_H^L &= ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H); \\ \Pi_H^R &= (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)); \\ \overline{\Pi}_H^L &= (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H); \\ \overline{\Pi}_H^R &= ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)). \end{aligned}$$

it is straightforward to show (see [9]) that they are idempotent and Π_H^L, Π_H^R satisfy the equalities

$$\Pi_H^L = id_H \wedge \lambda_H; \quad \Pi_H^R = \lambda_H \wedge id_H. \quad (5)$$

and then

$$\Pi_H^L \wedge \Pi_H^L = \Pi_H^L, \quad \Pi_H^R \wedge \Pi_H^R = \Pi_H^R. \quad (6)$$

Moreover, we have that

$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L; \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R; \quad \Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L; \quad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R; \quad (7)$$

$$\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L; \quad \overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R; \quad \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L; \quad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R. \quad (8)$$

Also it is easy to show the formulas

$$\Pi_H^L = \overline{\Pi}_H^R \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^L; \quad \Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R; \quad (9)$$

$$\Pi_H^L \circ \lambda_H = \Pi_H^L \circ \Pi_H^R = \lambda_H \circ \Pi_H^R; \quad \Pi_H^R \circ \lambda_H = \Pi_H^R \circ \Pi_H^L = \lambda_H \circ \Pi_H^L. \quad (10)$$

For the morphisms target and source we have the following identities:

$$\mu_H \circ (H \otimes \Pi_H^L) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H), \quad (11)$$

$$\mu_H \circ (\Pi_H^R \otimes H) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H), \quad (12)$$

$$\mu_H \circ (H \otimes \overline{\Pi}_H^L) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H), \quad (13)$$

$$\mu_H \circ (\overline{\Pi}_H^R \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes \delta_H), \quad (14)$$

$$(H \otimes \Pi_H^L) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H), \quad (15)$$

$$(\Pi_H^R \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)), \quad (16)$$

$$(\overline{\Pi}_H^L \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ ((\delta_H \circ \eta_H) \otimes H), \quad (17)$$

$$(H \otimes \overline{\Pi}_H^R) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)), \quad (18)$$

and

$$\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L) = \Pi_H^L \circ \mu_H, \quad (19)$$

$$\Pi_H^R \circ \mu_H \circ (\Pi_H^R \otimes H) = \Pi_H^R \circ \mu_H, \quad (20)$$

$$(H \otimes \Pi_H^L) \circ \delta_H \circ \Pi_H^L = \delta_H \circ \Pi_H^L, \quad (21)$$

$$(\Pi_H^R \otimes H) \circ \delta_H \circ \Pi_H^R = \delta_H \circ \Pi_H^R. \quad (22)$$

Definition 2.3. Let H be a weak Hopf algebra. We will say that a right H -comodule (A, ρ_A) is a right H -comodule algebra if it satisfies

$$\rho_A \circ \mu_A = \mu_{A \otimes H} \circ (\rho_A \otimes \rho_A)$$

and any of the following equivalent conditions hold:

- (b1) $(A \otimes \Pi_H^L) \circ \rho_A = (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ ((\rho_A \circ \eta_A) \otimes A).$
- (b2) $(A \otimes \overline{\Pi}_H^R) \circ \rho_A = (\mu_A \otimes H) \circ (A \otimes (\rho_A \circ \eta_A)).$
- (b3) $(A \otimes \Pi_H^L) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A.$
- (b4) $(A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A.$
- (b5) $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes \mu_H \otimes H) \circ (\rho_A \otimes \delta_H) \circ (\eta_A \otimes \eta_H).$
- (b6) $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\rho_A \otimes \delta_H) \circ (\eta_A \otimes \eta_H).$

If (A, ρ_A) is a right H -comodule algebra, the triple (A, H, Γ_A^H) is a right-right weak entwining structure (see [15]) where

$$\Gamma_A^H = (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \rho_A). \quad (23)$$

Therefore the following identities hold:

$$\Gamma_A^H \circ (H \otimes \mu_A) = (\mu_A \otimes H) \circ (A \otimes \Gamma_A^H) \circ (\Gamma_A^H \otimes A), \quad (24)$$

$$(A \otimes \delta_H) \circ \Gamma_A^H = (\Gamma_A^H \otimes H) \circ (H \otimes \Gamma_A^H) \circ (\delta_H \otimes A), \quad (25)$$

$$\Gamma_A^H \circ (H \otimes \eta_A) = (e_A \otimes H) \circ \delta_H, \quad (26)$$

$$(A \otimes \varepsilon_H) \circ \Gamma_A^H = \mu_A \circ (e_A \otimes A), \quad (27)$$

where

$$e_A = (A \otimes \varepsilon_H) \circ \Gamma_A^H \circ (H \otimes \eta_A). \quad (28)$$

We denote by $\mathcal{M}_A^H(\Gamma_A^H)$ the category of weak entwined modules, i.e., the objects M in \mathcal{C} together with two morphisms $\phi_M : M \otimes A \rightarrow A$ and $\rho_M : M \rightarrow M \otimes H$ such that (M, ϕ_M) is a right A -module, (M, ρ_M) is a right H -comodule and such that the following equality

$$\rho_M \circ \phi_M = (\phi_M \otimes H) \circ (M \otimes \Gamma_A^H) \circ (\rho_M \otimes A) \quad (29)$$

holds.

Then, if (A, ρ_A) is a right H -comodule algebra, (A, μ_A, ρ_A) is an object of $\mathcal{M}_A^H(\Gamma_A^H)$.

If (A, ρ_A) is a right H -comodule algebra, we define the subalgebra of coinvariants of A as the equalizer:

$$\begin{array}{ccccc} A_H & \xrightarrow{i_A} & A & \xrightleftharpoons[\zeta_A]{\rho_A} & A \otimes H \end{array}$$

where $\zeta_A = (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ ((\rho_A \circ \eta_A) \otimes A)$. Note that, by (b1), we have

$$\zeta_A = (A \otimes \Pi_H^L) \circ \rho_A. \quad (30)$$

and also, by (7) and (8),

$$A_H \xrightarrow{i_A} A \xrightleftharpoons[(A \otimes \overline{\Pi}_H^R) \circ \rho_A]{\rho_A} A \otimes H$$

is an equalizer diagram.

It is not difficult to see that $(A_H, \eta_{A_H}, \mu_{A_H})$ is an algebra, being η_{A_H} and μ_{A_H} the factorizations through the equalizer i_A of the morphisms η_A and $\mu_A \circ (i_A \otimes i_A)$, respectively.

For example, the weak Hopf algebra H is a right H -comodule algebra with right comodule structure giving by $\rho_H = \delta_H$ and subalgebra of coinvariants H_H , the image of the idempotent morphism Π_H^L , which we will denote by H_L .

Definition 2.4. Let H be a weak Hopf algebra and let (A, ρ_A) be a right H -comodule algebra. We define an integral as a morphism of right H -comodules $f : H \rightarrow A$. If moreover $f \circ \eta_H = \eta_A$ we will say that the integral is total.

An integral $f : H \rightarrow A$ is convolution invertible if there exists a morphism $f^{-1} : H \rightarrow A$ (called the convolution inverse of f) such that

- (c1) $f^{-1} \wedge f = e_A$.
- (c2) $f \wedge f^{-1} = (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes H)$.
- (c3) $f^{-1} \wedge f \wedge f^{-1} = f^{-1}$.

where e_A is the morphism defined in (28).

Trivially, the inverse is unique because if $h : H \rightarrow A$ satisfies (c1)-(c3),

$$h = h \wedge f \wedge h = h \wedge f \wedge f^{-1} = f^{-1} \wedge f \wedge f^{-1} = f^{-1}. \quad (31)$$

Moreover, using condition (c1), if f is an integral convolution invertible we get that $f \wedge f^{-1} \wedge f = f$.

Finally, when f is a total integral we can rewrite equality (c1) as $f^{-1} \wedge f = f \circ \Pi_H^R$ and (c2) as $f \wedge f^{-1} = f \circ \overline{\Pi}_H^L$.

Example 2.5. Let H be a weak Hopf algebra such that $\Pi_H^L = \overline{\Pi}_H^L$ (equivalently, $\Pi_H^R = \overline{\Pi}_H^R$). Then the identity id_H is a total integral convolution invertible with inverse λ_H . Note that this equality is always true in the Hopf algebra setting. In our case it holds, for example, if H is a cocommutative weak Hopf algebra.

Definition 2.6. Let H be a weak Hopf algebra and (A, ρ_A) a right H -comodule algebra. We say that $A_H \hookrightarrow A$ is a H -cleft extension if there exists an integral $f : H \rightarrow A$ convolution invertible and such that the morphism $f \wedge f^{-1}$ factorizes through the equalizer i_A . In what follows, the morphism f will be called a cleaving morphism associated to the H -cleft extension $A_H \hookrightarrow A$.

Proposition 2.7. Let H be a weak Hopf algebra and (A, ρ_A) a right H -comodule algebra such that $A_H \hookrightarrow A$ is a H -cleft extension with cleaving morphism f . Then the equality

$$\rho_A \circ f^{-1} = (f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H \quad (32)$$

holds.

Proof:

We define the morphisms: $r = \rho_A \circ f^{-1}$, $s = \rho_A \circ f$ and $t = (f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H$.

First of all, we show that $s \wedge r = s \wedge t$. Indeed:

$$\begin{aligned} & s \wedge r \\ &= \rho_A \circ (f \wedge f^{-1}) \\ &= (A \otimes \Pi_H^L) \circ \rho_A \circ (f \wedge f^{-1}) \\ &= (\mu_A \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L))) \circ (A \otimes c_{H,A} \otimes H) \circ ((\rho_A \circ f) \otimes (\rho_A \circ f^{-1})) \circ \delta_H \\ &= (\mu_A \otimes \Pi_H^L) \circ (A \otimes c_{H,A}) \circ ((\rho_A \circ f) \otimes f^{-1}) \circ \delta_H \end{aligned}$$

$$\begin{aligned}
&= (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ (((f \otimes (\mu_H \circ (H \otimes \lambda_H) \circ \delta_H)) \circ \delta_H) \otimes f^{-1}) \circ \delta_H \\
&= s \wedge t.
\end{aligned}$$

In the foregoing calculations, the first equality follows using that A is a right H -comodule algebra; the second one because $A_H \hookrightarrow A$ is H -cleft; in the third we use (19); the fourth relies on the equality

$$(\mu_A \otimes \mu_H) \circ (A \otimes c_{H,A} \otimes \Pi_H^L) \circ (\rho_A \otimes \rho_A) = (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \otimes (\rho_A \otimes A), \quad (33)$$

the fifth one is a consequence of the definition of Π_H^L ; finally, in the last one we use that f is an integral. Using similar techniques we obtain that $t \wedge s = r \wedge s$:

$$\begin{aligned}
&t \wedge s \\
&= (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (\lambda_H \otimes (f^{-1} \wedge f) \otimes H) \circ (H \otimes \delta_H) \circ \delta_H \\
&= (A \otimes \mu_H) \circ (A \otimes (\lambda_H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H)) \otimes H \circ (c_{H,A} \otimes H \otimes H) \circ (H \otimes (\rho_A \circ \eta_A) \otimes H) \circ \delta_H \\
&= (A \otimes \mu_H) \circ (A \otimes (\lambda_H \circ \mu_H \circ (H \otimes \overline{\Pi}_H^L)) \otimes H) \circ (c_{H,A} \otimes H \otimes H) \circ (H \otimes (\rho_A \circ \eta_A) \otimes H) \circ \delta_H \\
&= (A \otimes \mu_H) \circ (A \otimes (\mu_H \circ c_{H,H} \circ (\lambda_H \otimes \Pi_H^L)) \otimes H) \circ (c_{H,A} \otimes H \otimes H) \circ (H \otimes (\rho_A \circ \eta_A) \otimes H) \circ \delta_H \\
&= (A \otimes \mu_H) \circ ((\rho_A \circ \eta_A) \circ \Pi_H^R) \\
&= (A \otimes H \otimes (\varepsilon_H \circ \mu_H)) \circ (A \otimes c_{H,H} \otimes H) \circ (c_{H,A} \otimes \mu_H \otimes H) \circ (H \otimes (\rho_A \circ \eta_A) \otimes (\delta_H \circ \eta_H)) \\
&= (A \otimes H \otimes (\varepsilon_H \circ \mu_H)) \circ (A \otimes c_{H,H} \otimes H) \circ (c_{H,A} \otimes H \otimes H) \circ (H \otimes ((\rho_A \otimes H) \circ (\rho_A \circ \eta_A))) \\
&= \rho_A \circ (f^{-1} \wedge f) \\
&= r \wedge s.
\end{aligned}$$

In the previous calculations, the first equality follows because f is an integral; the second one by coassociativity of δ_H and the naturality of c ; in the third we use (13). The fourth equality relies on the antimultiplicativity of the antipode and (9); the fifth and the sixth ones follow by the definition of Π_H^R ; finally, in the seventh, eight and ninth equalities we use that A is a right H -comodule algebra.

On the other hand, under the conditions of this proposition we have that the equality

$$(f^{-1} \wedge f) \circ \mu_H = ((\varepsilon_H \circ \mu_H) \otimes (f^{-1} \wedge f)) \circ (H \otimes \delta_H) \quad (34)$$

holds. Indeed:

$$\begin{aligned}
&(f^{-1} \wedge f) \circ \mu_H \\
&= (A \otimes (\varepsilon_H \circ \mu_H \circ (H \otimes \mu_H))) \circ (c_{H,A} \otimes H \otimes H) \circ (H \otimes c_{H,A} \otimes H) \circ (H \otimes H \otimes (\rho_A \circ \eta_A)) \\
&= (A \otimes (((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H))) \circ (c_{H,A} \otimes H \otimes H) \circ (H \otimes c_{H,A} \otimes H) \circ (H \otimes H \otimes (\rho_A \circ \eta_A)) \\
&= (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\varepsilon_H \circ \mu_H) \otimes c_{H,A} \otimes H) \circ (H \otimes \delta_H \otimes \rho_A \circ \eta_A) \\
&= ((\varepsilon_H \circ \mu_H) \otimes (f^{-1} \wedge f)) \circ (H \otimes \delta_H),
\end{aligned}$$

where the first and the last equalities follow because f is an integral, the second one by (a2) of Definition 2.1 and the third one by the naturality of c .

Now we use the coassociativity of δ_H , the naturality of c , (34) and that f is a convolution invertible integral to get that $t \wedge s \wedge t = t$.

$$t \wedge s \wedge t$$

$$\begin{aligned}
&= (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ (A \otimes \mu_H \otimes A) \circ (c_{H,A} \otimes H \otimes A) \circ (\lambda_H \otimes (((f^{-1} \wedge f) \otimes \Pi_H^L) \circ \delta_H) \otimes A) \circ \\
&\quad (\delta_H \otimes f^{-1}) \circ \delta_H \\
&= (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ (A \otimes \mu_H \otimes A) \circ (c_{H,A} \otimes H \otimes A) \circ \\
&\quad (\lambda_H \otimes (((f^{-1} \wedge f) \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)) \otimes A) \circ (\delta_H \otimes f^{-1}) \circ \delta_H \\
&= (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ (A \otimes \mu_H \otimes A) \circ (c_{H,A} \otimes H \otimes A) \circ \\
&\quad (\lambda_H \otimes (((\varepsilon_H \circ \mu_H) \otimes (f^{-1} \wedge f)) \circ (H \otimes \delta_H)) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)) \otimes A) \circ (\delta_H \otimes f^{-1}) \circ \delta_H \\
&= c_{H,A} \circ ((\lambda_H \wedge \Pi_H^L) \otimes (f^{-1} \wedge f \wedge f^{-1})) \circ \delta_H \\
&= t.
\end{aligned}$$

Taking into account that $r \wedge s \wedge r = r$,

$$r = r \wedge s \wedge r = t \wedge s \wedge r = t \wedge s \wedge t = t$$

and we conclude the proof. \square

Proposition 2.8. *Let H be a cocommutative weak Hopf algebra and let (A, ρ_A) be a right H -comodule algebra. If there exists a convolution invertible integral $f : H \rightarrow A$, then $A_H \hookrightarrow A$ is an H -cleft extension.*

Proof:

Let f^{-1} be the convolution inverse of f . We have to show that $f \wedge f^{-1}$ factorizes through the equalizer i_A . Indeed:

$$\begin{aligned}
&\zeta_A \circ (f \wedge f^{-1}) \\
&= (A \otimes \Pi_H^L) \circ \rho_A \circ (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes H) \\
&= (A \otimes H \otimes (\varepsilon_H \circ \mu_H)) \circ (A \otimes ((\Pi_H^L \otimes H) \circ \delta_H) \otimes H) \circ ((\rho_A \circ \eta_A) \otimes H) \\
&= (A \otimes H \otimes (\varepsilon_H \circ \mu_H)) \circ (A \otimes ((\Pi_H^L \otimes H) \circ c_{H,H} \circ \delta_H) \otimes H) \circ ((\rho_A \circ \eta_A) \otimes H) \\
&= (A \otimes (\varepsilon_H \circ \mu_H) \otimes H) \circ (\rho_A \otimes c_{H,H}) \circ (((A \otimes \Pi_H^L) \circ \rho_A \circ \eta_A) \otimes H) \\
&= (A \otimes (\varepsilon_H \circ \mu_H) \otimes H) \circ (\rho_A \otimes c_{H,H}) \circ ((\rho_A \circ \eta_A) \otimes H) \\
&= (A \otimes H \otimes (\varepsilon_H \circ \mu_H)) \circ (A \otimes (c_{H,H} \circ \delta_H) \otimes H) \circ ((\rho_A \circ \eta_A) \otimes H) \\
&= (A \otimes H \otimes (\varepsilon_H \circ \mu_H)) \circ (A \otimes \delta_H \otimes H) \circ ((\rho_A \circ \eta_A) \otimes H) \\
&= \rho_A \circ (f \wedge f^{-1}).
\end{aligned}$$

In the foregoing calculations, the first and the last equalities follow by (c2) of Definition 2.4; the second, fourth and sixth ones use the condition of comodule for A ; in the third and seventh we use that H is cocommutative; finally the fifth one follows by (b3) of Definition 2.3. \square

2.9. Let H be a weak Hopf algebra and let (A, ρ_A) be a right H -comodule algebra. We want to point out the relation between the notion of H -cleft extension and the notion of weak H -cleft extension given by us in [2]. In [2] we introduce the set $Reg^{WR}(H, A)$ as the one whose elements are the morphisms

$h : H \rightarrow A$ such that there exists a morphism $h^{-1} : H \rightarrow A$, called the left weak inverse of h , such that

$$h^{-1} \wedge h = e_A \quad (35)$$

where e_A is the morphism defined in (28) for the right-right weak entwining structure Γ_A^H associated to (A, ρ_A) (see (23)).

Then, following the Definition 1.9 of [2], we say that $A_H \hookrightarrow A$ is a weak H -cleft extension if there exists a morphism $h : H \rightarrow A$ in $\text{Reg}^{WR}(H, A)$ of right H -comodules such that

$$\Gamma_A^H \circ (H \otimes h^{-1}) \circ \delta_H = \zeta_A \circ (e_A \wedge h^{-1}). \quad (36)$$

This definition of weak cleft extension is the one used in [25] where was proved that this kind of weak cleft extensions induce weak crossed products with a left invertible cocycle (see Definition 4.1 of [25]).

Also the definition of introduced in [2] is a generalization of the one given by Brzeziński [12] (see [16], [18] for the classical definitions in the Hopf algebra setting) in the context of entwined structures. Note that, while in the case of a cleft extension for an entwining structure h is required to be a comodule morphism and convolution invertible, here both conditions are replaced by weaker ones. Also, in [3] we proved that weak H -cleft extensions are exactly weak H -Galois extensions with normal basis (see Definition 1.8 and Theorem 2.11).

Note that if h is a morphism of right H -comodules

$$h \wedge e_A = \mu_A \circ (id_A \otimes e_A) \circ \rho_A \circ h = h$$

and if $g = e_A \wedge h^{-1}$ we have

$$g \wedge h = (e_A \wedge h^{-1}) \wedge h = e_A \wedge (h^{-1} \wedge h) = e_A \wedge e_A = e_A$$

and

$$e_A \wedge g = e_A \wedge (e_A \wedge h^{-1}) = (e_A \wedge e_A) \wedge h^{-1} = e_A \wedge h^{-1} = g.$$

Therefore, we can assume without loss of generality that $e_A \wedge h^{-1} = h^{-1}$ and (36) can be expressed as

$$\Gamma_A^H \circ (H \otimes h^{-1}) \circ \delta_H = \zeta_A \circ h^{-1}. \quad (37)$$

Moreover, By Remark 1.10 of [2], we know that if there exists $h \in \text{Reg}^{WR}(H, A)$ of right H -comodules

$$\Gamma_A^H = (\mu_A \otimes H) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((h^{-1} \otimes h) \circ \delta_H) \otimes A) \quad (38)$$

and the extension is weak cleft, by Proposition 1.12 of [2], we obtain that

$$q_A = \mu_A \circ (A \otimes h^{-1}) \circ \rho_A : A \rightarrow A$$

factors through i_A . Therefore, there exists a unique morphism $p_A : A \rightarrow A_H$ such that $q_A = i_A \circ p_A$. Then, $h \wedge h^{-1} = q_A \circ h$ and, as a consequence, $h \wedge h^{-1}$ admits a factorization through i_A .

Theorem 2.10. *Let H be a weak Hopf algebra and let (A, ρ_A) be a right H -comodule algebra. If there exists $h \in \text{Reg}^{WR}(H, A)$ of right H -comodules such that $e_A \wedge h^{-1} = h^{-1}$, the following assertions are equivalent:*

- (i) *The morphism $h \wedge h^{-1}$ factorizes through the equalizer i_A and h^{-1} satisfies (32).*
- (ii) *The equality (37) holds.*

Proof:

If (ii) holds, $A_H \hookrightarrow A$ is a weak H -cleft extension and then $h \wedge h^{-1}$ admits a factorization through i_A . The equality (32) follows in a similar way to the proof given in Proposition 2.7 using that $e_A \wedge h^{-1} = h^{-1}$.

Conversely, if we assume that (i) holds we have the following:

$$\begin{aligned} & \Gamma_A^H \circ (H \otimes h^{-1}) \circ \delta_H \\ &= (\mu_A \otimes H) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((h^{-1} \otimes h) \circ \delta_H) \otimes h^{-1}) \circ \delta_H \\ &= (\mu_A \otimes H) \circ (h^{-1} \otimes (\rho_A \circ (h \wedge h^{-1}))) \circ \delta_H \\ &= (\mu_A \otimes \Pi_H^L) \circ (h^{-1} \otimes (\rho_A \circ (h \wedge h^{-1}))) \circ \delta_H \\ &= ((\mu_A \circ (A \otimes \mu_A)) \otimes (\Pi_H^L \circ \mu_H)) \circ (h^{-1} \otimes h \otimes c_{H,A} \otimes H) \circ (H \otimes \delta_H \otimes (\rho_A \circ h^{-1})) \circ (H \otimes \delta_H) \circ \delta_H \end{aligned}$$

$$\begin{aligned}
&= (A \otimes \Pi_H^L) \circ \mu_{A \otimes H} \circ (e_A \otimes ((H \otimes ((h^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H)) \circ \delta_H)) \circ \delta_H \\
&= (\mu_A \otimes \Pi_H^L) \circ (e_A \otimes c_{H,A}) \circ (\delta_H \otimes h^{-1}) \circ \delta_H \\
&= (((A \otimes \varepsilon_H) \circ \Gamma_A^H) \otimes \Pi_H^L) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes h^{-1}) \circ \delta_H \\
&= (A \otimes (\varepsilon_H \circ \mu_H) \otimes \Pi_H^L) \circ (c_{H,A} \otimes c_{H,H}) \circ (H \otimes c_{H,A} \otimes H) \circ (\delta_H \otimes (\rho_A \circ h^{-1})) \circ \delta_H \\
&= (A \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L))) \circ (c_{H,A} \otimes H) \circ (H \otimes (\rho_A \circ h^{-1})) \circ \delta_H \\
&= (A \otimes (\Pi_H^L \circ \mu_H)) \circ (c_{H,A} \otimes H) \circ (H \otimes (\rho_A \circ h^{-1})) \circ \delta_H \\
&= (A \otimes (\Pi_H^L \circ \mu_H)) \circ (c_{H,A} \otimes H) \circ (H \otimes ((h^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H)) \circ \delta_H \\
&= (h^{-1} \otimes \Pi_H^L) \circ c_{H,H} \circ \delta_H
\end{aligned}$$

□

where the first equality follows by (38), the second one by the coassociativity of δ_H , the third one by the factorization of $h \wedge h^{-1}$ through i_A and the fourth one because A is a weak entwined module, δ_H is coassociative and h is a morphism of right H -comodules. In the fifth equality we used the coassociativity of δ_H and (32) and the sixth one is a consequence of the coassociativity of δ_H and the properties of Π_H^L . The seventh one follows by (27) and the eight one relies in the definition of Γ_A^H and the naturality of c . Using the naturality of c and (11) we obtain the ninth equality and the tenth one follows by (19). Finally, the eleventh one follows by (32) and the last one by the naturality of c and the properties of Π_H^L .

On the other hand,

$$\zeta_A \circ h^{-1} = (A \otimes \bar{\Pi}_H^R) \circ \rho_A \circ h^{-1} = (h^{-1} \otimes (\bar{\Pi}_H^R \circ \lambda_H)) \circ c_{H,H} \circ \delta_H = (h^{-1} \otimes \Pi_H^L) \circ c_{H,H} \circ \delta_H,$$

and the proof is complete.

As a corollary we have,

Corollary 2.11. *Let H be a weak Hopf algebra and let (A, ρ_A) be a right H -comodule algebra. If $A_H \hookrightarrow A$ is an H -cleft extension then it is a weak H -cleft extension.*

Proof:

If $A_H \hookrightarrow A$ is an H -cleft extension, there exists an integral $f : H \rightarrow A$ convolution invertible and such that the morphism $f \wedge f^{-1}$ factorizes through the equalizer i_A , being f^{-1} the convolution inverse of f . Then, $f \in \text{Reg}^{WR}(H, A)$, $e_A \wedge f^{-1} = f^{-1}$ and by Proposition 2.7, the equality (32) holds. Therefore, by the previous Theorem we obtain that $A_H \hookrightarrow A$ is a weak H -cleft extension.

□

Remark 2.12. As a consequence of Corollary 2.11, the results proved in [2] and [3] for weak H -cleft extensions can be applied for H -cleft extensions. For example, if $A_H \hookrightarrow A$ is an H -cleft extension with cleaving morphism f , the morphism $q_A = \mu_A \circ (A \otimes f^{-1}) \circ \rho_A$ factors through the equalizer i_A , i. e., there exists a morphism $p_A : A \rightarrow A_H$ such that $i_A \circ p_A = q_A$. Also, by Lemmas 3.9 and 3.11 of [4] we have the equalities:

$$\mu_A \circ (A \otimes e_A) \circ \rho_A = id_A, \quad (39)$$

$$\mu_A \circ (q_A \otimes f) \circ \rho_A = id_A, \quad (40)$$

$$\rho_A \circ \mu_A = (\mu_A \otimes H) \circ (q_A \otimes (\rho_A \circ \mu_A \circ (f \otimes A))) \circ (\rho_A \otimes A), \quad (41)$$

$$\mu_A \circ (i_A \otimes f) = \mu_A \circ (q_A \otimes A) \circ (\mu_A \otimes f) \circ (i_A \otimes (\rho_A \circ f)), \quad (42)$$

$$p_A \circ \mu_A \circ (i_A \otimes A) = \mu_{A_H} \circ (A_H \otimes p_A). \quad (43)$$

Definition 2.13. Let H be a weak Hopf algebra. Two H -cleft extensions $A_H \hookrightarrow A$ and $B_H \hookrightarrow B$ are equivalent (written $A_H \hookrightarrow A \sim B_H \hookrightarrow B$) if $A_H = B_H$ and there is a morphism of right H -comodule algebras $T : A \rightarrow B$ such that $T \circ i_A = i_B$.

Note that, if the H -cleft extensions $A_H \hookrightarrow A$ and $A_H \hookrightarrow B$ are equivalent and f is a cleaving morphism for $A_H \hookrightarrow A$, it is easy to show that $g = T \circ f$ is a cleaving morphism for $A_H \hookrightarrow B$ with $g^{-1} = T \circ f^{-1}$. Under these conditions, T is an isomorphism. If f is the cleaving morphism associated to $A_H \hookrightarrow A$, we define the morphisms

$$\begin{aligned}\gamma_A &= (p_A \otimes H) \circ \rho_A : A \rightarrow A_H \otimes H, \\ \chi_A &= \mu_A \circ (i_A \otimes f) : A_H \otimes H \rightarrow A\end{aligned}$$

and

$$\begin{aligned}\gamma_B &= (p_B \otimes H) \circ \rho_B : B \rightarrow A_H \otimes H, \\ \chi_B &= \mu_B \circ (i_B \otimes g) : A_H \otimes H \rightarrow B\end{aligned}$$

where p_A and p_B are the factorizations of $q_A = \mu_A \circ (A \otimes f^{-1}) \circ \rho_A$, $q_B = \mu_B \circ (A \otimes g^{-1}) \circ \rho_B$ and i_A , i_B the corresponding equalizer morphisms. Then,

$$\begin{aligned}\chi_B \circ \gamma_A &= \mu_B \circ ((i_B \circ p_A) \otimes g) \circ \rho_A = \mu_B \circ ((T \circ q_A) \otimes (T \circ f)) \circ \rho_A = T \circ \mu_A \circ (A \otimes (f^{-1} \wedge f)) \circ \rho_A \\ &= T \circ \mu_A \circ (A \otimes e_A) \circ \rho_A = T,\end{aligned}$$

and

$$i_B \circ p_B \circ T = \mu_B \circ (B \otimes (T \circ f^{-1})) \circ \rho_B \circ T = T \circ q_A = T \circ i_A \circ p_A = i_B \circ p_A.$$

If we define $T^{-1} : B \rightarrow A$ by $T^{-1} = \chi_A \circ \gamma_B$ we have:

$$\begin{aligned}T \circ T^{-1} &= T \circ \mu_A \circ ((i_A \circ p_B) \otimes f) \circ \rho_B = \mu_B \circ ((T \circ i_A \circ p_B) \otimes (T \circ f)) \otimes \rho_B \\ &= \mu_B \circ (B \otimes e_B) \circ \rho_B = id_B\end{aligned}$$

and, by a similar proof $T^{-1} \circ T = id_A$. Therefore, T is an isomorphism.

Obviously, " \sim " is an equivalence relation, and we will denote by $[B_H \hookrightarrow B]$ the isomorphisms class of the H -cleft extension $B_H \hookrightarrow B$.

It is a well known fact in the Hopf algebra setting that, if $A_H \hookrightarrow A$ is an H -cleft extension with convolution invertible integral f , the morphism $h = \mu_A \circ (f \otimes (f^{-1} \circ \eta_H))$ is a total integral convolution invertible. There is a similar result for weak Hopf algebras although we want to point out that the proof is very different, because in our case $\delta_H \circ \eta_H \neq \eta_H \otimes \eta_H$. Actually, in order to give the convolution inverse of this morphism we assume that the weak Hopf algebra is cocommutative. This hypothesis can be removed in the classical case because for Hopf algebras the morphisms Π_H^L , Π_H^R , $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ trivialize.

Proposition 2.14. *Let H be a weak Hopf algebra with invertible antipode. If $A_H \hookrightarrow A$ is an H -cleft extension with cleaving morphism f , then $h = \mu_A \circ (f \otimes (f^{-1} \circ \eta_H))$ is a total integral. Moreover, if H is cocommutative h is convolution invertible.*

Proof:

We define $h = \mu_A \circ (f \otimes (f^{-1} \circ \eta_H))$, where f^{-1} is the convolution inverse of f . Using Proposition 2.7, the properties of the antipode and (9),

$$\begin{aligned}& (A \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,A} \otimes H) \circ (H \otimes (\rho_A \circ f^{-1} \circ \eta_H)) \\ &= (A \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,A} \otimes H) \circ (H \otimes ((f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H \circ \eta_H)) \\ &= ((\varepsilon_H \circ \mu_H \circ c_{H,H}) \otimes f^{-1}) \circ (\lambda_H^{-1} \otimes (\delta_H \circ \eta_H)) \\ &= f^{-1} \circ \Pi_H^L \circ \lambda_H^{-1} \\ &= f^{-1} \circ \overline{\Pi}_H^R.\end{aligned}$$

As a consequence, using that f is an integral and A a comodule algebra, we obtain that

$$h = f \wedge (f^{-1} \circ \overline{\Pi}_H^R), \tag{44}$$

and then we get that h is total because, by (44) and (18),

$$h \circ \eta_H = (f \wedge (f^{-1} \circ \overline{\Pi}_H^R)) \circ \eta_H = (f \wedge f^{-1}) \circ \eta_H = \eta_A.$$

Moreover, h is an integral. Indeed:

$$\begin{aligned}
& \rho_A \circ h \\
&= \mu_{A \otimes H} \circ ((\rho_A \circ f) \otimes (\rho_A \circ f^{-1} \circ \eta_H)) \\
&= (A \otimes (\lambda_H \circ \lambda_H^{-1})) \circ \mu_{A \otimes H} \circ (((f \otimes H) \circ \delta_H) \otimes ((f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H \circ \eta_H)) \\
&= (\mu_A \otimes H) \circ (f \otimes (c_{H,A} \circ (\lambda_H \otimes f^{-1}) \circ (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes \lambda_H^{-1}))) \circ \delta_H \\
&= (\mu_A \otimes H) \circ (f \otimes (c_{H,A} \circ (\lambda_H \otimes f^{-1}) \circ (H \otimes \Pi_H^L) \circ \delta_H \circ \lambda_H^{-1})) \circ \delta_H \\
&= (\mu_A \otimes H) \circ (f \otimes ((f^{-1} \circ \Pi_H^L \circ \lambda_H^{-1}) \otimes H) \circ \delta_H) \circ \delta_H \\
&= (h \otimes H) \circ \delta_H
\end{aligned}$$

The first equality follows because A is a right H -comodule algebra, the second one uses that λ_H is an isomorphism, the third and the fifth ones are a consequence of the antimultiplicative property for λ_H , λ_H^{-1} and the naturality of c , the fourth one follows by (15) and, finally, the last one by the coassociativity of δ_H and (9).

Now we assume that H is cocommutative. We define $h^{-1} = \mu_A \circ ((f \circ \eta_H) \otimes f^{-1})$. Following a similar way to the one developed for h , it is not difficult to prove the equalities:

$$h^{-1} = (f \circ \Pi_H^R) \wedge f^{-1} \quad (45)$$

and

$$\mu_A \circ (f^{-1} \otimes (f \circ \eta_H)) = \mu_A \circ (f^{-1} \otimes (f \circ \Pi_H^R \circ \lambda_H)) \circ c_{H,H} \circ \delta_H. \quad (46)$$

As a consequence of the last equation, taking into account that H is cocommutative we obtain that

$$\mu_A \circ ((f^{-1} \circ \eta_H) \otimes (f \circ \eta_H)) = \eta_H. \quad (47)$$

We conclude the proof showing that h^{-1} is the convolution inverse of h . Condition (c2) follows because, by (47),

$$h \wedge h^{-1} = f \wedge f^{-1} = (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes H).$$

Moreover

$$\begin{aligned}
& h^{-1} \wedge h \\
&= \mu_A \circ (\mu_A \otimes A) \circ ((f \circ \eta_H) \otimes (f^{-1} \wedge f) \otimes (f^{-1} \circ \eta_H)) \\
&= \mu_A \circ (\mu_A \otimes (\varepsilon_H \circ \mu_H) \otimes A) \circ ((f \circ \eta_H) \otimes c_{H,A} \otimes H \otimes (f^{-1} \circ \eta_H)) \circ (H \otimes (\rho_A \circ \eta_A)) \\
&= \mu_A \circ (A \otimes (\varepsilon_H \circ \mu_H \circ c_{H,H} \circ (\overline{\Pi}_H^R \otimes H)) \otimes A) \circ ((\rho_A \circ f \circ \eta_H) \otimes H \otimes (f^{-1} \circ \eta_H)) \\
&= \mu_A \circ ((f \circ \Pi_H^R) \otimes (f^{-1} \circ \eta_H)) \\
&= h \circ \Pi_H^R \\
&= f^{-1} \wedge f \\
&= (A \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,A} \otimes H) \circ (H \otimes (\rho_A \circ \eta_A)).
\end{aligned}$$

In the foregoing calculations, the first equality follows by the definition of h and h^{-1} ; the second and the last ones because f is convolution invertible; the third because A is an H -comodule algebra; the fourth uses that f is an integral and (18); finally, the fifth and the sixth ones are consequence of the definition of the total integral h .

The proof for the condition (c3) for h follows a similar pattern and we leave the details to the reader.

□

Remark 2.15. As a consequence of the previous proposition, in the cocommutative setting we can assume that the integral is total.

In the following definition we recall the notion of weak left H -module algebra introduced in [6].

Definition 2.16. Let H be a weak Hopf algebra. We will say that A is a weak left H -module algebra if there exists a morphism $\varphi_A : H \otimes A \rightarrow A$ satisfying:

- (d1) $\varphi_A \circ (\eta_H \otimes A) = id_A$.
 - (d2) $\varphi_A \circ (H \otimes \mu_A) = \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)$.
 - (d3) $\varphi_A \circ (\mu_H \otimes \eta_A) = \varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A)))$.
- and any of the following equivalent conditions hold:
- (d4) $\varphi_A \circ (\Pi_H^L \otimes A) = \mu_A \circ ((\varphi_A \circ (H \otimes \eta_A)) \otimes A)$.
 - (d5) $\varphi_A \circ (\overline{\Pi}_H^L \otimes A) = \mu_A \circ c_{A,A} \circ ((\varphi_A \circ (H \otimes \eta_A)) \otimes A)$.
 - (d6) $\varphi_A \circ (\Pi_H^L \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$.
 - (d7) $\varphi_A \circ (\overline{\Pi}_H^L \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$.
 - (d8) $\varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A))) = ((\varphi_A \circ (H \otimes \eta_A)) \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H)$.
 - (d9) $\varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A))) = ((\varepsilon_H \circ \mu_H) \otimes (\varphi_A \circ (H \otimes \eta_A))) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$.

If we replace (d3) by

$$(d3-1) \quad \varphi_A \circ (\mu_H \otimes A) = \varphi_A \circ (H \otimes \varphi_A)$$

we will say that (A, φ_A) is a left H -module algebra.

Remark 2.17. Note that by (d4) and (d5) if the weak Hopf algebra is cocommutative the morphism $\varphi_A \circ (H \otimes \eta_A)$ factors through the center of A . Moreover, if H is a Hopf algebra and (A, φ_A) is a weak left H -module algebra, conditions (d4)-(d9) imply that $\varepsilon_H \otimes \eta_A = \varphi_A \circ (H \otimes \eta_A)$. As a consequence the equality (d3) is always true and φ_A is a weak action of H on A (see [7]).

Proposition 2.18. Let H be a cocommutative weak Hopf algebra. If $A_H \hookrightarrow A$ is an H -cleft extension with cleaving morphism f , the pair (A_H, φ_{A_H}) is a weak left H -module algebra, being φ_{A_H} the factorization of the morphism

$$\varphi_A = \mu_A \circ (A \otimes (\mu_A \circ c_{A,A})) \circ (((f \otimes f^{-1}) \circ \delta_H) \otimes i_A)$$

through the equalizer i_A .

Proof:

If $A_H \hookrightarrow A$ is a H -cleft extension, by Corollary 2.11, we have that $A_H \hookrightarrow A$ is a weak H -cleft extension, and, by Proposition 1.15 of [2], we know that φ_{A_H} factors through the equalizer i_A and satisfies (b2). Moreover it is not difficult to see that

$$\varphi_{A_H} = p_A \circ \mu_A \circ (f \otimes i_A), \tag{48}$$

and then (b1) holds.

As far as (b3),

$$\begin{aligned} & \varphi_{A_H} \circ (H \otimes (\varphi_{A_H} \circ (H \otimes \eta_{A_H}))) \\ &= p_A \circ \mu_A \circ (f \otimes (q_A \circ f)) \\ &= p_A \circ \mu_A \circ (f \otimes (f \wedge f^{-1})) \\ &= ((p_A \circ \mu_A) \otimes (\varepsilon_H \circ \mu_H)) \circ (f \otimes (\rho_A \circ \eta_A) \circ H) \\ &= (p_A \otimes (\varepsilon_H \circ \mu_H)) \circ (A \otimes \overline{\Pi}_H^R \otimes H) \circ ((\rho_A \circ f) \otimes H) \\ &= ((p_A \circ q_A \circ f) \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H) \\ &= ((p_A \circ (f \wedge f^{-1})) \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H) \\ &= (p_A \otimes (\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes \delta_H \otimes H) \\ &= (p_A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes \mu_H) \end{aligned}$$

$$= (p_A \circ (f \wedge f^{-1})) \circ \mu_H$$

$$= \varphi_{A_H} \circ (\mu_H \otimes \eta_{A_H}),$$

where the first equality follows by (48); the second because $q_A \circ f = f \wedge f^{-1}$; the third, sixth and eighth ones by (c2); in the fourth one we use that A is a right H -comodule algebra; in the fifth (14) and that f is an integral; the seventh equality follows because H is a weak Hopf algebra; finally, in the last one we use that $p_A \circ (f \wedge f^{-1}) = \varphi_{A_H} \circ (H \otimes \eta_{A_H})$.

It only remains to show one of the equivalent conditions (d4)-(d9). We get (d6):

$$\begin{aligned} & \varphi_{A_H} \circ (\Pi_H^L \otimes \eta_{A_H}) \\ &= p_A \circ (f \wedge f^{-1}) \circ \Pi_H^L \\ &= p_A \circ (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes \Pi_H^L) \\ &= p_A \circ (f \wedge f^{-1}) \\ &= \varphi_{A_H} \circ (H \otimes \eta_{A_H}). \end{aligned}$$

□

As a consequence of the previous result we have the following corollary.

Corollary 2.19. *In the conditions of Proposition 2.18,*

- (i) $\varphi_{A_H} = \varphi_{A_H} \circ (\overline{\Pi}_H^L \otimes A_H)$ if and only if $\mu_A \circ (f \otimes i_A) = \mu_A \circ c_{A,A} \circ (f \otimes i_A)$.
- (ii) $\varphi_{A_H} = \varphi_{A_H} \circ (\Pi_H^L \otimes A_H)$ if and only if $\mu_A \circ (f^{-1} \otimes i_A) = \mu_A \circ c_{A,A} \circ (f^{-1} \otimes i_A)$.

Proof:

We will show (i). Part (ii) is similar and we leave the details to the reader. Assume that $\varphi_{A_H} = \varphi_{A_H} \circ (\overline{\Pi}_H^L \otimes A_H)$. Taking into account (d5) and Proposition 2.18 we obtain that $\varphi_A = \mu_A \circ c_{A,A} \circ ((\varphi_A \circ (H \otimes \eta_A)) \otimes i_A)$. The result follows composing in this equality with $(H \otimes A_H \otimes f) \circ (H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H)$ on the right and with μ_A on the left. Indeed, using the definition of φ_A , that f is a convolution invertible integral, (b2), the properties of the equalizer i_A and that (A, ρ_A) is a right H -comodule algebra,

$$\begin{aligned} & \mu_A \circ (\varphi_A \otimes A) \circ (H \otimes ((i_A \otimes f) \circ c_{H,A_H})) \circ (\delta_H \otimes A_H) \\ &= \mu_A \circ (A \otimes \mu_A) \circ (f \otimes i_A \otimes f^{-1} \wedge f) \circ (H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H) \\ &= (\mu_A \otimes (\varepsilon_H \circ \mu_H)) \circ (f \otimes c_{H,A} \otimes H) \circ (\delta_H \otimes ((\mu_A \otimes H) \circ (i_A \otimes (\rho_A \circ \eta_A)))) \\ &= (A \otimes \varepsilon_H) \circ \mu_{A \otimes H} \circ ((\rho_A \circ f) \otimes ((A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ i_A)) \\ &= (A \otimes \varepsilon_H) \circ \rho_A \circ \mu_A \circ (f \otimes i_A) \\ &= \mu_A \circ (f \otimes i_A). \end{aligned}$$

On the other hand, by similar arguments,

$$\begin{aligned} & \mu_A \circ ((\mu_A \circ c_{A,A} \circ ((\varphi_A \circ (H \otimes \eta_A)) \otimes i_A) \otimes f) \circ (H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H)) \\ &= \mu_A \circ (i_A \otimes f \wedge f^{-1} \wedge f) \circ c_{H,A_H} \\ &= \mu_A \circ (i_A \otimes f) \circ c_{H,A_H} \end{aligned}$$

$$= \mu_A \circ c_{A,A} \circ (f \otimes i_A).$$

Conversely, by the hypothesis, (d5) and the equality $f \wedge f^{-1} = \varphi_A \circ (H \otimes \eta_A)$,

$$\begin{aligned} & i_A \circ \varphi_{A_H} \\ &= \mu_A \circ ((\mu_A \circ (f \otimes i_A)) \otimes f^{-1}) \circ (H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H) \\ &= \mu_A \circ (i_A \otimes f \wedge f^{-1}) \circ c_{H,A_H} \\ &= \mu_A \circ c_{A,A} \circ ((\varphi_A \circ (H \otimes \eta_A)) \circ i_A) \\ &= \varphi_A \circ (\overline{\Pi}_H^L \otimes i_A) \\ &= i_A \circ \varphi_{A_H} \circ (\overline{\Pi}_H^L \otimes A_H), \\ & \text{and then } \varphi_{A_H} = \varphi_{A_H} \circ (\overline{\Pi}_H^L \otimes A_H). \end{aligned}$$

□

3. CROSSED SYSTEMS FOR WEAK HOPF ALGEBRAS

In the first part of this section we generalize the theory of crossed systems over a Hopf algebra given by Doi in [17] to the weak setting. Taking into account the theory developed in the previous section, being H a cocommutative weak Hopf algebra, we will obtain a bijective correspondence between the isomorphisms classes of H -cleft extensions $[A_H \hookrightarrow A]$ and the equivalence classes of crossed systems for H over A_H .

By Propositions 1.4, 1.6 and 1.8 of [6], if H is a weak Hopf algebra, the morphisms

$$\Omega_H^L = ((\varepsilon_H \circ \mu_H) \otimes H \otimes H) \circ \delta_{H \otimes H} : H \otimes H \rightarrow H \otimes H \quad (49)$$

$$\Omega_H^R = (H \otimes H \otimes (\varepsilon_H \circ \mu_H)) \circ \delta_{H \otimes H} : H \otimes H \rightarrow H \otimes H \quad (50)$$

are idempotent and satisfy that $\mu_H = \mu_H \circ \Omega_H^L = \mu_H \circ \Omega_H^R$. Also, Ω_H^L is a morphism of left and right H -modules for the usual regular actions because

$$\Omega_H^L = ((\mu_H \circ (H \otimes \Pi_H^L)) \otimes H) \circ (H \otimes \delta_H) = (H \otimes (\mu_H \circ (\overline{\Pi}_H^R \otimes H))) \circ ((c_{H,H} \circ \delta_H) \otimes H). \quad (51)$$

Moreover, if H is cocommutative, $\Omega_H^L = \Omega_H^R$ and we denote it by Ω_H^2 . Then, we have

$$(\Omega_H^2 \otimes H \otimes H) \circ \delta_{H \otimes H} = (H \otimes H \otimes \Omega_H^2) \circ \delta_{H \otimes H} = \delta_{H \otimes H} \circ \Omega_H^2. \quad (52)$$

Following Definition 1.18 of [6] we have:

Definition 3.1. Let H be a cocommutative weak Hopf algebra and (A, φ_A) be a weak left H -module algebra. We define $Reg_{\varphi_A}(H, A)$, as the set of morphisms $h : H \rightarrow A$ such that there exists a morphism $h^{-1} : H \rightarrow A$ (the convolution inverse of h) satisfying the following equalities:

- (e1) $h \wedge h^{-1} = h^{-1} \wedge h = u_1$,
- (e2) $h \wedge h^{-1} \wedge h = h$,
- (e3) $h^{-1} \wedge h \wedge h^{-1} = h^{-1}$,

where $u_1 = \varphi_A \circ (H \otimes \eta_A)$.

In a similar way, $Reg_{\varphi_A}(H \otimes H, A)$ is the set of morphisms $\sigma : H \otimes H \rightarrow A$ such that there exists a morphism $\sigma^{-1} : H \otimes H \rightarrow A$ satisfying:

- (f1) $\sigma \wedge \sigma^{-1} = \sigma^{-1} \wedge \sigma = u_2$
- (f2) $\sigma \wedge \sigma^{-1} \wedge \sigma = \sigma$.
- (f3) $\sigma^{-1} \wedge \sigma \wedge \sigma^{-1} = \sigma^{-1}$.

where $u_2 = \varphi_A \circ (H \otimes u_1)$.

Note that, by (d3) of Definition 2.16,

$$u_2 = u_1 \circ \mu_H \quad (53)$$

and if $\sigma \in \text{Reg}_{\varphi_A}(H \otimes H, A)$, in [6], we prove that

$$\sigma = \sigma \circ \Omega_H^2 \quad (54)$$

Analogously, $\sigma^{-1} = \sigma^{-1} \circ \Omega_H^2$.

Also, by Proposition 1.19 of [6] we know that, if H is a weak Hopf algebra and (A, φ_A) a weak left H -module algebra such that there exists $h : H \rightarrow A$ satisfying that:

$$h \wedge h^{-1} = h^{-1} \wedge h = u_1, \quad h \wedge h^{-1} \wedge h = h, \quad h^{-1} \wedge h \wedge h^{-1} = h^{-1},$$

the following equalities are equivalent

$$h \circ \eta_H = \eta_A, \quad (55)$$

$$h \circ \Pi_H^L = u_1, \quad (56)$$

$$h \circ \overline{\Pi}_H^L = u_1. \quad (57)$$

If λ_H is an isomorphism they are equivalent to

$$h \circ \Pi_H^R = u_1 \circ \lambda_H, \quad (58)$$

and

$$h \circ \overline{\Pi}_H^R = u_1 \circ \lambda_H^{-1}. \quad (59)$$

In a similar way, it is possible to see that, if $\sigma : H \otimes H \rightarrow A$ is a morphism such that

$$\sigma \wedge \sigma^{-1} = \sigma^{-1} \wedge \sigma = u_2, \quad \sigma \wedge \sigma^{-1} \wedge \sigma = \sigma, \quad \sigma^{-1} \wedge \sigma \wedge \sigma^{-1} = \sigma^{-1},$$

the following equalities are equivalent:

$$\sigma \circ (\eta_H \otimes H) = u_1, \quad (60)$$

$$\sigma \circ (\Pi_H^L \otimes H) \circ \delta_H = u_1, \quad (61)$$

$$\sigma \circ c_{H,H} \circ (H \otimes \overline{\Pi}_H^L) \circ \delta_H = u_1. \quad (62)$$

If the antipode λ_H is an isomorphism (60)-(62) are equivalent to

$$\sigma \circ (\Pi_H^R \otimes \lambda_H) \circ \delta_H = u_1 \circ \lambda_H, \quad (63)$$

and

$$\sigma \circ c_{H,H} \circ (\lambda_H^{-1} \otimes \overline{\Pi}_H^R) \circ \delta_H = u_1 \circ \lambda_H^{-1}. \quad (64)$$

Finally, the following assertions are equivalent:

$$\sigma \circ (H \otimes \eta_H) = u_1, \quad (65)$$

$$\sigma \circ (H \otimes \Pi_H^R) \circ \delta_H = u_1, \quad (66)$$

$$\sigma \circ c_{H,H} \circ (\overline{\Pi}_H^R \otimes H) \circ \delta_H = u_1. \quad (67)$$

If the antipode λ_H is an isomorphism (65)-(67) are equivalent to

$$\sigma \circ (\lambda_H \otimes \Pi_H^L) \circ \delta_H = u_1 \circ \lambda_H, \quad (68)$$

and

$$\sigma \circ c_{H,H} \circ (\overline{\Pi}_H^L \otimes \lambda_H^{-1}) \circ \delta_H = u_1 \circ \lambda_H^{-1}. \quad (69)$$

Proposition 3.2. *Let H be a weak Hopf algebra and (A, φ_A) be a weak left H -module algebra. If there exists $h : H \rightarrow A$ satisfying the following equalities:*

$$h \wedge h^{-1} = h^{-1} \wedge h = u_1, \quad h \wedge h^{-1} \wedge h = h, \quad h^{-1} \wedge h \wedge h^{-1} = h^{-1},$$

we have that $h \circ \eta_H = \eta_A$ if and only if $h^{-1} \circ \eta_H = \eta_A$.

Proof:

If $h \circ \eta_H = \eta_A$, by (56) and (15), we have

$$h^{-1} \circ \eta_H = (h^{-1} \wedge u_1) \circ \eta_H = (h^{-1} \wedge (h \circ \Pi_H^L)) \circ \eta_H = u_1 \circ \eta_H = \eta_A.$$

Conversely, if $h^{-1} \circ \eta_H = \eta_A$, by similar arguments,

$$h \circ \eta_H = (h \wedge u_1) \circ \eta_H = (h \wedge (h^{-1} \circ \Pi_H^L)) \circ \eta_H = u_1 \circ \eta_H = \eta_A.$$

□

Definition 3.3. Let H be a weak Hopf algebra, (A, φ_A) a weak left H -module algebra and $\sigma : H \otimes H \rightarrow A$ a morphism satisfying (f1)-(f3). We say that (φ_A, σ) is a crossed system for H over A if the following conditions hold:

- (g1) $\mu_A \circ (A \otimes \varphi_A) \circ (\sigma \otimes \mu_H \otimes A) \circ (\delta_{H \otimes H} \otimes A)$
 $= \mu_A \circ ((\varphi_A \circ (H \otimes \varphi_A)) \otimes A) \circ (H \otimes H \otimes c_{A,A}) \circ (H \otimes H \otimes \sigma \otimes A) \circ (\delta_{H \otimes H} \otimes A).$
- (g2) $((\varphi_A \circ (H \otimes \sigma)) \circ (\Omega_H^L \otimes H)) \wedge (\sigma \circ (H \otimes \mu_H)) = ((\sigma \otimes \varepsilon_H) \circ (H \otimes \Omega_H^L) \wedge (\sigma \circ (\mu_H \otimes H))).$
- (g3) $\sigma \circ (H \otimes \eta_H) = \sigma \circ (\eta_H \otimes H) = \varphi_A \circ (H \otimes \eta_A).$

Note that equalities (g1) and (g3) are the same that the ones given by Doi in [17], while (g2) is slightly different on account of it includes the idempotent morphism Ω_H^2 . Anyway, when H is a Hopf algebra, $\Omega_H^2 = id_{H \otimes H}$ and our definition is a generalization to the weak Hopf algebra setting of the one introduced in [17].

Also, it is clear that our condition (f1) over σ implies that it is left invertible in the sense of Definition 4.1 of [25]. In any case, to obtain the main results of this paper and a good cohomological interpretation, we need the right invertibility, that is (f1) and (f2), (f3).

Moreover, if the weak Hopf algebra H is cocommutative the morphism σ is in $Reg_{\varphi_A}(H \otimes H, A)$ and we can remove the morphism Ω_H^2 in condition (g2). Indeed, by (54) and using that $\Omega_H^L = \Omega_H^2$ is a morphism of left and right H -modules we obtain that (g2) is equivalent to

$$(\varphi_A \circ (H \otimes \sigma)) \wedge (\sigma \circ (H \otimes \mu_H)) = ((\sigma \otimes \varepsilon_H)) \wedge (\sigma \circ (\mu_H \otimes H)). \quad (70)$$

Moreover, (70) is equivalent to

$$\begin{aligned} \mu_A \circ ((\varphi_A \circ (H \otimes \sigma^{-1})) \otimes \sigma) \circ (H \otimes H \otimes c_{H,H} \otimes H) \circ (H \otimes H \otimes H \otimes c_{H,H}) \circ (\delta_{H \otimes H} \otimes H) \\ = \mu_A \circ (\sigma \otimes \sigma^{-1}) \circ (H \otimes \mu_H \otimes \mu_H \otimes H) \circ \delta_{H \otimes H \otimes H} \end{aligned} \quad (71)$$

and to

$$\mu_A \circ (\sigma^{-1} \otimes (\varphi_A \circ (H \otimes \sigma))) \circ (\delta_{H \otimes H} \otimes H) = (\sigma \circ (\mu_H \otimes H)) \wedge (\sigma^{-1} \circ (H \otimes \mu_H)) \quad (72)$$

Two crossed systems for H over A , (φ_A, σ) and (ϕ_A, τ) are said to be equivalent, denoted by

$$(\varphi_A, \sigma) \approx (\phi_A, \tau),$$

if $\varphi_A \circ (H \otimes \eta_A) = \phi_A \circ (H \otimes \eta_A)$ and there exists h in $Reg_{\varphi_A}(H, A) \cap Reg_{\phi_A}(H, A)$ with $h \circ \eta_H = \eta_A$ and such that

$$\varphi_A = \mu_A \circ (\mu_A \otimes A) \circ (h \otimes \phi_A \otimes h^{-1}) \circ (\delta_H \otimes c_{H,A}) \circ (\delta_H \otimes A), \quad (73)$$

$$\sigma = \mu_A \circ (\mu_A \otimes h^{-1}) \circ (\mu_A \otimes \tau \otimes \mu_H) \circ (h \otimes \phi_A \otimes \delta_{H \otimes H}) \circ (\delta_H \otimes h \otimes H \otimes H) \circ \delta_{H \otimes H}. \quad (74)$$

Proposition 3.4. Let H be a cocommutative weak Hopf algebra. Then \approx is an equivalence relation.

Proof:

Let (φ_A, σ) be a crossed system. The morphism u_1 is in $Reg_{\varphi_A}(H, A)$ with inverse $u_1^{-1} = u_1$ and satisfies that $u_1 \circ \eta_H = \eta_A$. Moreover, using that (A, φ_A) is a weak left H -module algebra,

$$\begin{aligned} \mu_A \circ (\mu_A \otimes A) \circ (u_1 \otimes \varphi_A \otimes u_1^{-1}) \circ (\delta_H \otimes c_{H,A}) \circ (\delta_H \otimes A) \\ = \mu_A \circ (\mu_A \otimes A) \circ ((\varphi_A \circ (H \otimes \eta_A)) \otimes \varphi_A \otimes (\varphi_A \circ (H \otimes \eta_A))) \circ (\delta_H \otimes c_{H,A}) \circ (\delta_H \otimes A) \\ = \mu_A \circ (\varphi_A \otimes (\varphi_A \circ (H \otimes \eta_A))) \circ (\delta_H \otimes c_{H,A}) \circ (\delta_H \otimes A) \\ = \varphi_A, \end{aligned}$$

and we get (73).

As far as (74), using that (A, φ_A) is a weak left H -module algebra and taking into account that σ is in $Reg_{\varphi_A}(H \otimes H, A)$,

$$\begin{aligned} \mu_A \circ (\mu_A \otimes u_1^{-1}) \circ (\mu_A \otimes \sigma \otimes \mu_H) \circ (u_1 \otimes \varphi_A \otimes \delta_{H \otimes H}) \circ (\delta_H \otimes u_1 \otimes H \otimes H) \circ \delta_{H \otimes H} \\ = \mu_A \circ ((\mu_A \circ (\varphi_A \circ (H \otimes \eta_A))) \otimes \varphi_A) \otimes (\sigma \wedge (\varphi_A \circ (\mu_H \otimes \eta_A))) \circ (\delta_H \otimes (\varphi_A \circ (H \otimes \eta_A))) \otimes H \otimes H \circ \delta_{H \otimes H} \\ = \mu_A \circ ((\varphi_A \circ (\mu_H \otimes \eta_A)) \otimes (\sigma \wedge \sigma^{-1} \wedge \sigma)) \circ \delta_{H \otimes H} \\ = \sigma \wedge \sigma^{-1} \wedge \sigma \end{aligned}$$

$$= \sigma,$$

and the relation is reflexive.

In order to get that \approx is symmetrical, assume that $(\varphi_A, \sigma) \approx (\phi_A, \tau)$. Let h be the morphism in $\text{Reg}_{\varphi_A}(H, A) \cap \text{Reg}_{\phi_A}(H, A)$ satisfying (73) and (74) and such that $h \circ \eta_H = \eta_A$. Then the inverse h^{-1} is in $\text{Reg}_{\varphi_A}(H, A) \cap \text{Reg}_{\phi_A}(H, A)$ and by Proposition 3.2 we obtain that $h^{-1} \circ \eta_H = \eta_A$. Moreover,

$$\begin{aligned} & \mu_A \circ (\mu_A \otimes A) \circ (h^{-1} \otimes \varphi_A \otimes h) \circ (\delta_H \otimes c_{H,A}) \circ (\delta_H \otimes A) \\ &= \mu_A \circ (\mu_A \otimes A) \circ ((h^{-1} \wedge h) \otimes \phi_A \otimes (h^{-1} \wedge h)) \circ (\delta_H \otimes c_{H,A}) \circ (\delta_H \otimes A) \\ &= \mu_A \circ (\mu_A \otimes A) \circ ((\phi_A \circ (H \otimes \eta_A)) \otimes \phi_A \otimes (\phi_A \circ (H \otimes \eta_A))) \circ (\delta_H \otimes c_{H,A}) \circ (\delta_H \otimes A) \\ &= \mu_A \circ (\phi_A \otimes (\phi_A \circ (H \otimes \eta_A))) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) \\ &= \phi_A \end{aligned}$$

using that $(\varphi_A, \sigma) \approx (\psi_A, \tau)$, (e1) and that (A, ψ_A) is a weak left H -module algebra.

In a similar way we obtain (74) and the relation is symmetrical.

Finally we show the transitivity. Assume that $(\varphi_A, \sigma) \approx (\phi_A, \tau)$ and $(\phi_A, \tau) \approx (\chi_A, \gamma)$ with morphisms h in $\text{Reg}_{\varphi_A}(H, A) \cap \text{Reg}_{\phi_A}(H, A)$ and g in $\text{Reg}_{\phi_A}(H, A) \cap \text{Reg}_{\chi_A}(H, A)$, respectively. Then, the convolution product $h \wedge g$ is in $\text{Reg}_{\varphi_A}(H, A) \cap \text{Reg}_{\chi_A}(H, A)$ and $(h \wedge g) \circ \eta_H = \eta_A$. Indeed:

$$\begin{aligned} & (h \wedge g) \circ \eta_H \\ &= (h \wedge (g \circ \Pi_H^L)) \circ \eta_H \\ &= (h \wedge g^{-1} \wedge g) \circ \eta_H \\ &= (h \wedge h^{-1} \wedge h) \circ \eta_H \\ &= h \circ \eta_H \\ &= \eta_A \end{aligned}$$

In the previous calculations, the first equality follows by (15), the second one by (56); in the third we use that $g^{-1} \wedge g = h^{-1} \wedge h$; the fourth relies on (e2) and the last one follows by the definition of h .

The proof for the conditions (73) and (74) follows a similar pattern to the well-know in the classical case, and we leave the details to the reader. \square

Remark 3.5. We have given the detailed calculus for the above Proposition in order to illustrate the differences when working with weak Hopf algebras. Note that the proof is trivial in the classical case: if H is a Hopf algebra, the relation is reflexive using the morphism $h = \varepsilon_H \otimes \eta_A$, and it is easy to get that it is symmetrical because $h \wedge h^{-1} = h^{-1} \wedge h = \varepsilon_H \otimes \eta_A$. Obviously, these equalities are not true for weak Hopf algebras.

Proposition 3.6. *Let H be a cocommutative weak Hopf algebra, (A, φ_A) a weak left H -module algebra and $\sigma \in \text{Reg}_{\varphi_A}(H \otimes H, A)$. The following assertions hold.*

- (i) $\sigma \circ (\eta_H \otimes H) = u_1 \Leftrightarrow \sigma^{-1} \circ (\eta_H \otimes H) = u_1.$
- (ii) $\sigma \circ (H \otimes \eta_H) = u_1 \Leftrightarrow \sigma^{-1} \circ (H \otimes \eta_H) = u_1.$

Proof:

We prove (i). The prove for (ii) is similar and we leave the details to the reader.

$$\begin{aligned} & \sigma^{-1} \circ (\eta_H \otimes H) \\ &= (\sigma^{-1} \wedge u_2) \circ (\eta_H \otimes H) \end{aligned}$$

$$\begin{aligned}
&= \mu_A \circ (\sigma^{-1} \otimes (u_1 \circ \mu_H)) \circ \delta_{H \otimes H} \circ (\eta_H \otimes H) \\
&= \mu_A \circ ((\sigma^{-1} \circ c_{H,H}) \otimes (u_1 \circ \mu_H)) \circ (H \otimes (\delta_H \circ \eta_H) \otimes H) \circ \delta_H \\
&= \mu_A \circ ((\sigma^{-1} \circ c_{H,H}) \otimes u_1) \circ (((\bar{\Pi}_H^L \otimes H) \circ \delta_H) \otimes H) \circ \delta_H \\
&= \mu_A \circ ((\sigma^{-1} \circ c_{H,H} \circ (H \otimes \bar{\Pi}_H^L) \circ \delta_H) \otimes (\sigma \circ (\Pi_H^L \otimes H) \circ \delta_H)) \circ \delta_H \\
&= \mu_A \circ ((\sigma^{-1} \circ c_{H,H}) \otimes \sigma) \circ (H \otimes (((\bar{\Pi}_H^L \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)) \otimes H) \circ (H \otimes \delta_H) \circ \delta_H \\
&= \mu_A \circ ((\sigma^{-1} \circ c_{H,H}) \otimes \sigma) \circ (H \otimes ((H \otimes ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H))) \\
&\quad \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H) \otimes H)) \otimes H) \circ (H \otimes \delta_H) \circ \delta_H \\
&= \mu_A \circ ((\sigma^{-1} \circ c_{H,H}) \otimes \sigma) \circ (H \otimes ((H \otimes (((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \circ \delta_H) \otimes H))) \\
&\quad \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H) \otimes H)) \otimes H) \circ (H \otimes (c_{H,H} \circ \delta_H)) \circ \delta_H \\
&= ((\mu_A \circ (\sigma^{-1} \otimes \sigma)) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes c_{H,H} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes ((\delta_H \otimes H) \circ \delta_H)) \\
&= ((\sigma^{-1} \wedge \sigma)) \otimes (\varepsilon_H \circ \mu_H)) \circ \delta_{H \otimes H} \circ (\eta_H \otimes H) \\
&= ((u_1 \circ \mu_H)) \otimes (\varepsilon_H \circ \mu_H)) \circ \delta_{H \otimes H} \circ (\eta_H \otimes H) \\
&= (u_1 \otimes \varepsilon_H) \circ \delta_H \circ \mu_H \circ (\eta_H \otimes H) \\
&= u_1,
\end{aligned}$$

where the first and the twelfth equalities follow by the properties of σ , the second one by the definition of u_2 , the third, tenth and eleventh ones by the naturality of c , the fourth one by (17), the fifth one by the coassociativity of δ_H and (61), the sixth one by the coassociativity of δ_H and (15), the seventh one by the definition of $\bar{\Pi}_H^L$ and the associativity of μ_H , the eighth one by (a3) of Definition 2.1, the ninth one by the cocommutativity of H , the thirteenth one by (a1) of Definition 2.1 and the last one by the unit-counit properties.

The proof for the converse is the same changing σ by σ^{-1} .

□

3.7. The equalities (g1), (g2) and (g3) of Definition 3.3 have a clear meaning in the theory of weak crossed products introduced in [4] and [19]. The full details can be found in Section 2 of [6]. In this point we give a brief resume adapted to our setting, i.e., with some changes in the notation.

Let H be a weak Hopf algebra, (A, φ_A) a weak left H -module algebra and $\sigma : H \otimes H \rightarrow A$ a morphism. We define the morphisms

$$\psi_H^A : H \otimes A \rightarrow A \otimes H, \quad \sigma_H^A : H \otimes H \rightarrow A \otimes H,$$

by

$$\psi_H^A = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) \quad (75)$$

and

$$\sigma_H^A = (\sigma \otimes \mu_H) \circ \delta_{H \otimes H}. \quad (76)$$

Then, ψ_H^A satisfies

$$(\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\psi_H^A \otimes A) = \psi_H^A \circ (H \otimes \mu_A). \quad (77)$$

As a consequence of (77), the morphism $\nabla_{A \otimes H} : A \otimes H \rightarrow A \otimes H$ defined by

$$\nabla_{A \otimes H} = (\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (A \otimes H \otimes \eta_A) \quad (78)$$

is idempotent. Moreover, $\nabla_{A \otimes H}$ satisfies that

$$\nabla_{A \otimes H} \circ (\mu_A \otimes H) = (\mu_A \otimes H) \circ (A \otimes \nabla_{A \otimes H}),$$

that is, $\nabla_{A \otimes H}$ is a left A -module morphism (see Lemma 3.1 of [19]) for the regular action $\varphi_{A \otimes H} = \mu_A \otimes H$. With $A \times H$, $i_{A \otimes H} : A \times H \rightarrow A \otimes H$ and $p_{A \otimes H} : A \otimes H \rightarrow A \times H$ we denote the object, the injection and the projection associated to the factorization of $\nabla_{A \otimes H}$. Moreover, if ψ_H^A satisfies (77), the following identities hold

$$(\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\nabla_{A \otimes H} \otimes A) = (\mu_A \otimes H) \circ (A \otimes \psi_H^A) = \nabla_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \psi_H^A) \quad (79)$$

$$\nabla_{A \otimes H} \circ \psi_H^A = \psi_H^A \quad (80)$$

and, by the naturality of c , we have

$$\nabla_{A \otimes H} = ((\mu_A \circ (A \otimes u_1)) \otimes H) \circ (A \otimes \delta_H) \quad (81)$$

and this implies that $\nabla_{A \otimes H}$ is a morphism of right H -comodules for $\rho_{A \otimes H} = A \otimes \delta_H$.

Also, in Propositions 2.7 and 2.8 of [6] we can find the proof of the equalities:

$$\nabla_{A \otimes H} = ((\mu_A \circ (A \otimes u_1)) \otimes H) \circ (A \otimes \delta_H), \quad (82)$$

$$\mu_A \circ (u_1 \otimes \varphi_A) \circ (\delta_H \otimes A) = \varphi_A, \quad (83)$$

$$(\mu_A \otimes H) \circ (u_1 \otimes \psi_H^A) \circ (\delta_H \otimes A) = \psi_H^A, \quad (84)$$

$$(A \otimes \varepsilon_H) \circ \psi_H^A \circ (H \otimes \eta_A) = u_1, \quad (85)$$

$$(\mu_A \otimes H) \circ (u_1 \otimes c_{H,A}) \circ (\delta_H \otimes A) = (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ ((\psi_H^A \circ (H \otimes \eta_A)) \otimes A), \quad (86)$$

$$(A \otimes \varepsilon_H) \circ \nabla_{A \otimes H} = \mu_A \circ (A \otimes u_1). \quad (87)$$

$$(A \otimes \delta_H) \circ \nabla_{A \otimes H} = (\nabla_{A \otimes H} \otimes H) \circ (H \otimes \delta_H). \quad (88)$$

$$(A \otimes \delta_H) \circ \sigma_H^A = (\sigma_H^A \otimes \mu_H) \circ \delta_{H \otimes H}. \quad (89)$$

Moreover, if H is a cocommutative and $\sigma \in \text{Reg}_{\varphi_A}(H \otimes H, A)$, the morphism σ_H^A satisfies the following identities:

$$\sigma_H^A \circ \Omega_H^2 = \sigma_H^A, \quad (90)$$

$$\nabla_{A \otimes H} \circ \sigma_H^A = \sigma_H^A, \quad (91)$$

$$(A \otimes \varepsilon_H) \circ \sigma_H^A = \sigma. \quad (92)$$

If we consider the quadruple $\mathbb{A}_H = (A, H, \psi_H^A, \sigma_H^A)$, where H a cocommutative weak Hopf algebra, (A, φ_A) is a weak left H -module algebra and $\sigma \in \text{Reg}_{\varphi_A}(H \otimes H, A)$, we say that \mathbb{A}_H satisfies the twisted condition if

$$(\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\sigma_H^A \otimes A) = (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \psi_H^A) \quad (93)$$

and the cocycle condition holds if

$$(\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\sigma_H^A \otimes H) = (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \sigma_H^A). \quad (94)$$

Note that, if $\mathbb{A}_H = (A, H, \psi_H^A, \sigma_H^A)$ satisfies the twisted condition, in Proposition 3.4 of [19], we proved that the following equalities hold:

$$(\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \nabla_{A \otimes H}) = \nabla_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H), \quad (95)$$

$$\nabla_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\nabla_{A \otimes H} \otimes H) = \nabla_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma_H^A). \quad (96)$$

Then, by (91), we obtain

$$(\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \nabla_{A \otimes H}) = (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H), \quad (97)$$

$$(\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\nabla_{A \otimes H} \otimes H) = (\mu_A \otimes H) \circ (A \otimes \sigma_H^A). \quad (98)$$

For the product defined by

$$\mu_{A \otimes_\sigma H} = (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes \psi_H^A \otimes H), \quad (99)$$

if the twisted and the cocycle conditions hold, we obtain that it is associative and normalized with respect to $\nabla_{A \otimes H}$ (i.e. $\nabla_{A \otimes H} \circ \mu_{A \otimes_\sigma H} = \mu_{A \otimes_\sigma H} = \mu_{A \otimes_\sigma H} \circ (\nabla_{A \otimes H} \otimes \nabla_{A \otimes H})$). Due to the normality condition,

$$\mu_{A \times_\sigma H} = p_{A \otimes H} \circ \mu_{A \otimes H} \circ (i_{A \otimes H} \otimes i_{A \otimes H}), \quad (100)$$

is associative as well (Proposition 3.7 of [19]). Hence if $\mathbb{A}_H = (A, H, \psi_H^A, \sigma_H^A)$ satisfies (93) and (94) we say that $A \otimes_\sigma H = (A \otimes H, \mu_{A \otimes_\sigma H})$ is a weak crossed product.

If \mathbb{A}_H satisfies

$$(\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \nu) = \nabla_{A \otimes H} \circ (\eta_A \otimes H), \quad (101)$$

$$(\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\nu \otimes H) = \nabla_{A \otimes H} \circ (\eta_A \otimes H), \quad (102)$$

$$(\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\nu \otimes A) = \beta_\nu, \quad (103)$$

for $\nu : K \rightarrow A \otimes H$ and

$$\beta_\nu = (\mu_A \otimes H) \circ (A \otimes \nu) : A \rightarrow A \otimes H, \quad (104)$$

by Theorem 2.2 of [6], we obtain that ν is a preunit (see (73) of [6] for the definition in this setting) for the product $\mu_{A \otimes_\sigma H}$ defined in (99). Therefore $A \times H$ is an algebra with the product defined in (100) and unit $\eta_{A \times_\sigma H} = p_{A \otimes H} \circ \nu$. In what follows we denote this algebra by $A \times_\sigma H$.

Following Definition 2.11 of [6], we say that σ satisfies the twisted condition if

$$\mu_A \circ ((\varphi_A \circ (H \otimes \varphi_A)) \otimes A) \circ (H \otimes H \otimes c_{A,A}) \circ (((H \otimes H \otimes \sigma) \circ \delta_{H \otimes H}) \otimes A) = \mu_A \circ (A \otimes \varphi_A) \circ (\sigma_H^A \otimes A), \quad (105)$$

and if

$$\mu_A \circ (A \otimes \sigma) \circ (\sigma_H^A \otimes H) = \mu_A \circ (A \otimes \sigma) \circ (\psi_H^A \otimes H) \circ (H \otimes \sigma_H^A) \quad (106)$$

holds, we will say that σ satisfies the 2-cocycle condition.

It is a trivial calculus to prove that (g1) of Definition 3.3 is equivalent to (105) and also (g2) is equivalent to (106). Moreover, by Theorem 2.13 of [6], we know that σ satisfies the twisted condition (105) if and only if \mathbb{A}_H satisfies the twisted condition (93), and, by Theorem 2.14 [6], we also know that σ satisfies the 2-cocycle condition (106) if and only if \mathbb{A}_H satisfies the cocycle condition (94).

On the other hand, (g3) of Definition 3.3 is exactly the normal condition introduced in Definition 2.16 of [6] and also, by Theorem 2.18 and Corollary 2.19 of [6], we have that $\nu = \nabla_{A \otimes H} \circ (\eta_A \otimes \eta_H)$ is a preunit for the weak crossed product associated to \mathbb{A}_H if and only if (g3) holds.

Therefore, if (φ_A, σ) is a crossed system for H over A , we have that $A \otimes_\sigma H$ is a weak crossed product, with preunit $\nu = \nabla_{A \otimes H} \circ (\eta_A \otimes \eta_H)$. Conversely, if the pair (φ_A, σ) is such that $A \otimes_\sigma H$ is a weak crossed product, with preunit $\nu = \nabla_{A \otimes H} \circ (\eta_A \otimes \eta_H)$ and normalized with respect to $\nabla_{A \otimes H}$, we obtain that (φ_A, σ) is a crossed system for H over A (see Corollary 2.20 of [6]).

Note that, if (φ_A, σ) and (ϕ_A, τ) are equivalent crossed systems for H over A , the equality

$$u_1 = \varphi_A \circ (H \otimes \eta_A) = \phi_A \circ (H \otimes \eta_A) = u'_1$$

holds and then the associated idempotent morphisms are the same because, by (81),

$$\nabla'_{A \otimes H} = ((\mu_A \circ (A \otimes u'_1)) \otimes H) \circ (A \otimes \delta_H) = ((\mu_A \circ (A \otimes u_1)) \otimes H) \circ (A \otimes \delta_H) = \nabla_{A \otimes H}. \quad (107)$$

Therefore, they define an algebra structure over the same object $A \times H$.

In the following result we characterize the crossed products where φ_A is an H -module structure.

Theorem 3.8. *Let H be a cocommutative weak Hopf algebra, (A, φ_A) a weak left H -module algebra and $\sigma \in \text{Reg}_{\varphi_A}(H \otimes H, A)$ satisfying the twisted condition (105). The following assertions are equivalent:*

- (i) (A, φ_A) is a left H -module algebra.
- (ii) The morphism σ factors through the center of A .

Proof:

Let (A, φ_A) be a left H -module algebra. We define $\gamma_\sigma : A \otimes H \otimes H \rightarrow A$ as

$$\gamma_\sigma = \mu_A \circ ((\mu_A \circ c_{A,A}) \otimes A) \circ (A \otimes ((\sigma \otimes \sigma^{-1}) \circ \delta_{H \otimes H})).$$

Then,

$$\gamma_\sigma = \mu_A \circ (A \otimes u_2) \quad (108)$$

because

$$\begin{aligned} & \gamma_\sigma \\ &= \mu_A \circ ((\mu_A \circ c_{A,A}) \otimes A) \circ (A \otimes (((\sigma \wedge u_2) \otimes \sigma^{-1}) \circ \delta_{H \otimes H})) \\ &= \mu_A \circ ((\mu_A \circ (A \otimes \mu_A)) \circ (A \otimes (c_{A,A} \circ (A \otimes u_1)))) \otimes A \circ (c_{A,A} \otimes H \otimes A) \circ (A \otimes \sigma_H^A \otimes \sigma^{-1}) \circ (A \otimes \delta_{H \otimes H}) \\ &= \mu_A \circ ((\mu_A \circ (A \otimes (\varphi_A \circ (\Pi_H^L \otimes A) \circ c_{A,H}))) \circ (c_{A,A} \otimes H)) \otimes A \circ (A \otimes \sigma_H^A \otimes \sigma^{-1}) \circ (A \otimes \delta_{H \otimes H}) \end{aligned}$$

$$\begin{aligned}
&= \mu_A \circ ((\mu_A \circ (A \otimes (A \otimes (\varphi_A \circ c_{A,H} \circ (A \otimes (\mu_H \circ (H \otimes \lambda_H) \circ (\mu_H \otimes \mu_H) \circ \delta_{H \otimes H})))))) \otimes A) \\
&\quad \circ (c_{A,A} \otimes H \otimes H \otimes A) \circ (A \otimes \sigma \otimes H \otimes H \otimes A) \circ (A \otimes \delta_{H \otimes H} \otimes \sigma^{-1}) \circ (A \otimes \delta_{H \otimes H}) \\
&= \mu_A \circ ((\mu_A \circ (A \otimes \varphi_A) \circ (\sigma_H^A \otimes A) \circ (H \otimes H \otimes (\varphi_A \circ c_{A,H} \circ (A \otimes \lambda_H)))) \circ (H \otimes c_{A,H} \otimes \mu_H) \\
&\quad \circ (c_{A,H} \otimes H \otimes H \otimes H) \circ (A \otimes \delta_{H \otimes H})) \otimes \sigma^{-1} \circ (A \otimes \delta_{H \otimes H}) \\
&= \mu_A \circ ((\mu_A \circ ((\varphi_A \circ (H \otimes \varphi_A)) \otimes A) \circ (H \otimes H \otimes c_{A,A}) \circ (((H \otimes H \otimes \sigma) \circ \delta_{H \otimes H}) \otimes A) \\
&\quad \circ (H \otimes H \otimes (\varphi_A \circ c_{A,H} \circ (A \otimes \lambda_H))) \circ (H \otimes c_{A,H} \otimes \mu_H) \circ (c_{A,H} \otimes H \otimes H \otimes H) \circ (A \otimes \delta_{H \otimes H})) \otimes \sigma^{-1}) \circ \\
&\quad (A \otimes \delta_{H \otimes H}) \\
&= \mu_A \circ (((\varphi_A \circ (\mu_H \otimes A) \circ (H \otimes \mu_H \otimes A)) \circ (H \otimes H \otimes c_{A,H})) \otimes \mu_A) \circ (H \otimes c_{A,H} \otimes (c_{A,H} \circ (\sigma \otimes (\lambda_H \circ \mu_H))) \\
&\quad \circ (A \otimes c_{H,H} \otimes H) \circ ((c_{H,H} \circ \delta_H) \otimes (c_{H,H} \circ \delta_H))) \otimes A) \circ (c_{A,H} \otimes H \otimes H \otimes H \otimes \sigma^{-1}) \\
&\quad \circ (H \otimes \delta_{H \otimes H} \otimes H \otimes H) \circ (A \otimes \delta_{H \otimes H}) \\
&= \mu_A \circ (((\varphi_A \circ (\mu_H \otimes A)) \circ (H \otimes c_{A,H})) \otimes A) \circ (c_{A,H} \otimes \lambda_H \otimes A) \circ (A \otimes ((\mu_H \otimes \mu_H) \circ \delta_{H \otimes H}) \otimes (\sigma \wedge \sigma^{-1})) \\
&\quad \circ (A \otimes \delta_{H \otimes H}) \\
&= \mu_A \circ ((\varphi_A \circ c_{A,H} \circ (A \otimes (\Pi_H^L \circ \mu_H))) \otimes u_2) \circ (A \otimes \delta_{H \otimes H}) \\
&= \mu_A \circ ((\varphi_A \circ (\overline{\Pi}_H^L \otimes A) \circ c_{A,H}) \otimes u_1) \circ (A \otimes (\delta_H \circ \mu_H)) \\
&= \mu_A \circ ((\mu_A \circ c_{A,A} \circ (u_1 \otimes A) \circ c_{A,H}) \otimes u_1) \circ (A \otimes (\delta_H \circ \mu_H)) \\
&= \mu_A \circ (A \otimes (u_1 \wedge u_1)) \circ (A \otimes \mu_H) \\
&= \mu_A \circ (A \otimes u_2)
\end{aligned}$$

where the first and the ninth equalities follow by (f1)-(f3), the second one by (53) and the naturality of c , the third one by the associativity of μ_A and by (d4) of Definition 2.16, the fourth one by the definition of Π_H^L , the fifth and the eight ones by the naturality of c , the coassociativity of δ_H and (d3-1) of Definition 2.16. The sixth one is a consequence of the twisted condition, the seventh follows by the naturality of c and the coassociativity and cocommutativity of δ_H , the tenth one by the cocommutativity of δ_H , the eleventh one by (d5) of Definition 2.16, the twelfth one by the naturality of c and finally, the last one, by (d2) of Definition 2.16.

Therefore, σ factors through the center of A because

$$\begin{aligned}
&\mu_A \circ (A \otimes \sigma) \\
&= \mu_A \circ (A \otimes (u_2 \wedge \sigma)) \\
&= \mu_A \circ ((\mu_A \circ (A \otimes u_2)) \otimes \sigma) \circ (A \otimes \delta_{H \otimes H}) \\
&= \mu_A \circ (\gamma_\sigma \otimes \sigma) \circ (A \otimes \delta_{H \otimes H}) \\
&= \mu_A \circ ((\mu_A \circ ((\mu_A \circ c_{A,A}) \otimes A) \circ (A \otimes ((\sigma \otimes \sigma^{-1}) \circ \delta_{H \otimes H}))) \otimes \sigma) \circ (A \otimes \delta_{H \otimes H})
\end{aligned}$$

$$\begin{aligned}
&= \mu_A \circ ((\mu_A \circ c_{A,A}) \otimes A) \circ (A \otimes ((\sigma \otimes (\sigma^{-1} \wedge \sigma)) \circ \delta_{H \otimes H})) \\
&= \mu_A \circ ((\mu_A \circ c_{A,A}) \otimes A) \circ (A \otimes ((\sigma \otimes u_2) \circ \delta_{H \otimes H})) \\
&= \mu_A \circ (A \otimes (\varphi_A \circ c_{A,H} \circ (A \otimes \overline{\Pi}_H^L))) \circ (c_{A,A} \otimes H) \circ (A \otimes \sigma_H^A) \\
&= \mu_A \circ (A \otimes (\varphi_A \circ (\Pi_H^L \otimes A) \circ c_{A,H})) \circ (c_{A,A} \otimes H) \circ (A \otimes \sigma_H^A) \\
&= \mu_A \circ (A \otimes (\mu_A \circ (u_1 \otimes A) \circ c_{A,H})) \circ (c_{A,A} \otimes H) \circ (A \otimes \sigma_H^A) \\
&= \mu_A \circ ((\sigma \wedge u_2) \otimes A) \circ (H \otimes c_{A,H}) \circ (c_{A,H} \otimes H) \\
&= \mu_A \circ c_{A,A} \circ (A \otimes \sigma)
\end{aligned}$$

where the first, the sixth and the eleventh equalities follow by (f1)-(f3), the second one by the associativity of μ_A , the third one by (108), the fourth one by the definition of γ_σ , the fifth one by the naturality of c , the coassociativity of δ_H and the associativity of μ_A , the seventh and the ninth ones by (d5) of Definition 2.16, the eight one by the cocommutativity of δ_H , the tenth one by the naturality of c and the associativity of μ_A .

Conversely, assume that the morphism σ factors through the center of A . Then, (A, φ_A) is a left H -module algebra because

$$\begin{aligned}
&\varphi_A \circ (H \otimes \varphi_A) \\
&= \varphi_A \circ (H \otimes (\mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes \eta_A \otimes A))) \\
&= \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes ((\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes \eta_A \otimes A))) \\
&= \mu_A \circ (u_2 \otimes (\varphi_A \circ (H \otimes \varphi_A))) \circ (\delta_{H \otimes H} \otimes A) \\
&= \mu_A \circ ((\sigma^{-1} \wedge \sigma) \otimes (\varphi_A \circ (H \otimes \varphi_A))) \circ (\delta_{H \otimes H} \otimes A) \\
&= \mu_A \circ (\sigma^{-1} \otimes (\mu_A \circ (\sigma \otimes (\varphi_A \circ (H \otimes \varphi_A))) \circ (\delta_{H \otimes H} \otimes A))) \circ (\delta_{H \otimes H} \otimes A) \\
&= \mu_A \circ (\sigma^{-1} \otimes (\mu_A \circ c_{A,A} \circ (\sigma \otimes (\varphi_A \circ (H \otimes \varphi_A))) \circ (\delta_{H \otimes H} \otimes A))) \circ (\delta_{H \otimes H} \otimes A) \\
&= \mu_A \circ (\sigma^{-1} \otimes (\mu_A \circ ((\varphi_A \circ (H \otimes \varphi_A)) \otimes A) \circ (H \otimes H \otimes c_{A,A}) \circ (((H \otimes H \otimes \sigma) \circ \delta_{H \otimes H}) \otimes A)) \circ (\delta_{H \otimes H} \otimes A)) \\
&= \mu_A \circ (\sigma^{-1} \otimes (\mu_A \circ (A \otimes \varphi_A) \circ (\sigma_H^A \otimes A))) \circ (\delta_{H \otimes H} \otimes A) \\
&= \mu_A \circ ((\sigma^{-1} \wedge \sigma) \otimes (\varphi_A \circ (\mu_H \otimes A))) \circ (\delta_{H \otimes H} \otimes A) \\
&= \mu_A \circ (u_2 \otimes (\varphi_A \circ (\mu_H \otimes A))) \circ (\delta_{H \otimes H} \otimes A) \\
&= \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ ((\delta_H \circ \mu_H) \otimes \eta_A \otimes A) \\
&= \varphi_A \circ (\mu_H \otimes A)
\end{aligned}$$

where the first, the second and the twelfth equalities follows by (d2) of Definition 2.16, the third one by the naturality of c and (d3) of Definition 2.16, the fourth and the tenth ones by (f1)-(f3), the fifth and the ninth ones by the naturality of c and the coassociativity of δ_H , the sixth one by the cocommutativity of δ_H and the factorization of σ through $Z(A)$, the seventh one by the naturality of c and the cocommutativity of H , the eight one by the twisted condition (105) and, finally, the eleventh one by (a1) of Definition 2.1.

□

Corollary 3.9. *Let H be a cocommutative weak Hopf algebra, (A, φ_A) a weak left H -module algebra. The following assertions are equivalent:*

- (i) (A, φ_A) is a left H -module algebra.
- (ii) (φ_A, u_2) is a crossed system for H over A .

Proof:

(i) \Rightarrow (ii) Trivially, the morphism u_2 is in $\text{Reg}_{\varphi_A}(H \otimes H, A)$ and satisfies (g2) and (g3). In order to get (g1), using the definition of u_2 , that H is a weak Hopf algebra, and that (A, φ_A) is a left H -module algebra,

$$\begin{aligned}
 & \mu_A \circ (A \otimes \varphi_A) \circ (u_2 \otimes \mu_H \otimes A) \circ (\delta_{H \otimes H} \otimes A) \\
 &= \mu_A \circ ((\varphi_A \circ (H \otimes \eta_A)) \otimes \varphi_A) \circ ((\delta_H \circ \mu_H) \otimes A) \\
 &= \varphi_A \circ (\mu_H \otimes A) \\
 &= \mu_A \circ (\varphi_A \otimes A) \circ (H \otimes c_{A,A}) \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A)) \otimes A) \circ ((\delta_H \circ \mu_H) \otimes A) \\
 &= \mu_A \circ ((\varphi_A \circ (H \otimes \varphi_A)) \otimes A) \circ (H \otimes H \otimes c_{A,A}) \circ (H \otimes H \otimes u_2 \otimes A) \circ (\delta_{H \otimes H} \otimes A).
 \end{aligned}$$

As far as (ii) \Rightarrow (i), note that by cocommutativity of H we have that $\Pi_H^L = \overline{\Pi}_H^L$ and u_2 factors through the center of A . Applying Theorem 3.8 we get that (A, φ_A) is a left H -module algebra.

□

Remark 3.10. In the conditions of Corollary 3.9, if (A, φ_A) is a left H -module algebra, we have that for the crossed system (φ_A, u_2)

$$\sigma_H^A = (u_1 \otimes H) \circ \delta_H \circ \mu_H. \quad (109)$$

Then, the associated crossed product defined in (99) is

$$\mu_{A \otimes u_2 H} = \nabla_{A \otimes H} \circ (\mu_A \otimes \mu_H) \circ (A \otimes \psi_H^A \otimes H). \quad (110)$$

and therefore

$$\mu_{A \times_{u_2} H} = p_{A \otimes H} \circ (\mu_A \otimes \mu_H) \circ (A \otimes \psi_H^A \otimes H) \circ (i_{A \otimes H} \otimes i_{A \otimes H}). \quad (111)$$

In this case we say that the weak crossed product is smash.

On the other hand, if for a weak left H -module algebra the equality

$$\varphi_A = \varphi_A \circ (\Pi_H^L \otimes A), \quad (112)$$

holds, by (d4) of Definition 2.16, we obtain that

$$\begin{aligned}
 & \mu_{A \times_{\sigma} H} \\
 &= p_{A \otimes H} \circ (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes c_{H,A} \otimes H) \circ ((\nabla_{A \otimes H} \circ i_{A \otimes H}) \otimes i_{A \otimes H}) \\
 &= p_{A \otimes H} \circ (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes c_{H,A} \otimes H) \circ (i_{A \otimes H} \otimes i_{A \otimes H}).
 \end{aligned}$$

In this case the weak crossed product is called twisted.

Proposition 3.11. *Let H be a cocommutative weak Hopf algebra and (φ_A, σ) a crossed system for H over A . Then, the algebra $A \times_{\sigma} H$ is a right H -comodule algebra for the coaction*

$$\rho_{A \times_{\sigma} H} = (p_{A \otimes H} \otimes H) \circ (A \otimes \delta_H) \circ i_{A \otimes H}.$$

Moreover, $(A \times_{\sigma} H)_H = A$.

Proof:

By Proposition 3.2 of [6], we obtain that $A \times_{\sigma} H$ is a right H -comodule algebra for the coaction

$$\rho_{A \times_{\sigma} H} = (p_{A \otimes H} \otimes H) \circ (A \otimes \delta_H) \circ i_{A \otimes H}.$$

Moreover,

$$\begin{array}{ccccc} A & \xrightarrow{i_{A \times_\sigma H}} & A \times H & \xrightarrow{\rho_{A \times_\sigma H}} & A \times H \otimes H \\ & & & \searrow & \\ & & & (A \times H \otimes \Pi_H^L) \circ \rho_{A \times_\sigma H} & \end{array}$$

is an equalizer diagram, where

$$i_{A \times_\sigma H} = p_{A \otimes H} \circ (A \otimes \eta_H).$$

Indeed, using that $\nabla_{A \otimes H}$ is a morphism of right H -comodules (see (81)) and (15) we have:

$$\begin{aligned} & \rho_{A \times_\sigma H} \circ i_{A \times_\sigma H} \\ &= (p_{A \otimes H} \otimes H) \circ (A \otimes \delta_H) \circ \nabla_{A \otimes H} \circ (A \otimes \eta_H) \\ &= (p_{A \otimes H} \otimes \Pi_H^L) \circ (A \otimes (\delta_H \circ \eta_H)) \\ &= (p_{A \otimes H} \otimes (\Pi_H^L \circ \Pi_H^L)) \circ (A \otimes (\delta_H \circ \eta_H)) \\ &= (A \times H \otimes \Pi_H^L) \circ \rho_{A \times_\sigma H} \circ i_{A \times_\sigma H}. \end{aligned}$$

Moreover, if $g : Q \rightarrow A \times H$ is a morphism such that $\rho_{A \times_\sigma H} \circ g = (A \times H \otimes \Pi_H^L) \circ \rho_{A \times_\sigma H} \circ g$ we obtain

$$\begin{aligned} & (A \otimes \delta_H) \circ i_{A \otimes H} \circ g \\ &= (\nabla_{A \otimes H} \otimes H) \circ (A \otimes \delta_H) \circ i_{A \otimes H} \circ g \\ &= (i_{A \otimes H} \otimes H) \circ \rho_{A \times_\sigma H} \circ g \\ &= (i_{A \otimes H} \otimes \Pi_H^L) \circ \rho_{A \times_\sigma H} \circ g \\ &= (A \otimes H \otimes \Pi_H^L) \circ (A \otimes \delta_H) \circ i_{A \otimes H} \circ g, \end{aligned}$$

and then

$$i_{A \otimes H} \circ g = (A \otimes \Pi_H^L) \circ i_{A \otimes H} \circ g. \quad (113)$$

Now we will show that $h = (A \otimes \varepsilon_H) \circ i_{A \otimes H} \circ g$ is the unique morphism such that $i_{A \otimes H} \circ h = g$.

First note that, by (81), we have

$$\begin{aligned} h &= (A \otimes \varepsilon_H) \circ i_{A \otimes H} \circ g = (A \otimes \varepsilon_H) \circ \nabla_{A \otimes H} \circ (A \otimes \Pi_H^L) \circ i_{A \otimes H} \circ g \\ &= \mu_A \circ (A \otimes u_1) \circ i_{A \otimes H} \circ g, \end{aligned}$$

and the equality

$$\nabla_{A \otimes H} \circ ((\mu_A \circ (A \otimes u_1)) \otimes \eta_H) = (A \otimes \Pi_H^L) \circ \nabla_{A \otimes H} \quad (114)$$

holds because

$$\begin{aligned} & \nabla_{A \otimes H} \circ ((\mu_A \circ (A \otimes u_1)) \otimes \eta_H) \\ &= (\mu_A \otimes H) \circ (A \otimes (\nabla_{A \otimes H} \circ (u_1 \otimes \eta_H))) \\ &= (\mu_A \otimes H) \circ (A \otimes (\psi_H^A \circ (\eta_H \otimes u_1))) \\ &= (\mu_A \otimes H) \circ (A \otimes (((u_1 \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H))) \\ &= (\mu_A \otimes H) \circ (A \otimes ((u_1 \otimes \Pi_H^L) \circ \delta_H)) \\ &= (A \otimes \Pi_H^L) \circ \nabla_{A \otimes H} \end{aligned}$$

In the foregoing calculations, the first equality follows using that $\nabla_{A \otimes H}$ is a morphism of left A -modules; the second one is a consequence of the definition of $\nabla_{A \otimes H}$; the third follows by (d3) of the definition of weak left H -module algebra; the fourth one by (15), and finally, the last one by (81).

Therefore,

$$i_{A \times_\sigma H} \circ h = p_{A \otimes H} \circ ((\mu_A \circ (A \otimes u_1)) \circ i_{A \otimes H} \circ g) \otimes \eta_H = p_{A \otimes H} \circ (A \otimes \Pi_H^L) \circ i_{A \otimes H} \circ g = g$$

Finally, if $r : Q \rightarrow A$ is a morphism such that $i_{A \times H} \circ r = g$, we have

$$r = (A \otimes \varepsilon_H) \circ \nabla_{A \otimes H} \circ (r \otimes \eta_H) = (A \otimes \varepsilon_H) \circ i_{A \otimes H} \circ g = h$$

and we conclude the proof. \square

In the following proposition we establish the relation between crossed systems and H -cleft extensions.

Proposition 3.12. *Let H be a cocommutative weak Hopf algebra and (φ_A, σ) a crossed system for H over A . Then $A \hookrightarrow A \times_\sigma H$ is an H -cleft extension.*

Proof:

The morphism $f = p_{A \otimes H} \circ (\eta_A \otimes H) : H \rightarrow A \times_\sigma H$ is a total integral. Obviously, $f \circ \eta_H = \eta_{A \times_\sigma H}$. Moreover, using that $\nabla_{A \otimes H}$ is a morphism of right H -comodules we get that f is an integral because:

$$\begin{aligned} & \rho_{A \times_\sigma H} \circ f \\ &= (p_{A \otimes H} \otimes H) \circ (A \otimes \delta_H) \circ \nabla_{A \otimes H} \circ (\eta_A \otimes H) \\ &= (f \otimes H) \circ \delta_H. \end{aligned}$$

We define $f^{-1} = p_{A \otimes H} \circ (\sigma^{-1} \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \lambda_H) \otimes H) \circ \delta_H$. We will show that f^{-1} is the convolution inverse of f . First note that (c1) holds:

$$\begin{aligned} & f^{-1} \wedge f \\ &= p_{A \otimes H} \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes ((u_1 \otimes H) \circ \delta_H) \otimes H) \circ (((\sigma^{-1} \otimes H) \circ (H \otimes c_{H,H})) \circ ((\delta_H \circ \lambda_H) \otimes H) \circ \delta_H) \otimes H \\ &= p_{A \otimes H} \circ ((\mu_A \circ (A \otimes (u_2 \wedge \sigma))) \otimes \mu_H) \circ (\sigma^{-1} \otimes \delta_{H \otimes H}) \circ (((H \otimes c_{H,H}) \circ ((\delta_H \circ \lambda_H) \otimes H) \circ \delta_H) \otimes H) \circ \delta_H \\ &= p_{A \otimes H} \circ (\mu_A \otimes H) \circ (\sigma^{-1} \otimes \sigma_H^A) \circ (((H \otimes c_{H,H}) \circ ((\delta_H \circ \lambda_H) \otimes H) \circ \delta_H) \otimes H) \circ \delta_H \\ &= p_{A \otimes H} \circ ((\sigma^{-1} \wedge \sigma) \otimes \mu_H) \circ \delta_{H \otimes H} \circ (\lambda_H \otimes H) \circ \delta_H \\ &= p_{A \otimes H} \circ ((u_1 \circ \mu_H) \otimes \mu_H) \circ \delta_{H \otimes H} \circ (\lambda_H \otimes H) \circ \delta_H \\ &= p_{A \otimes H} \circ (u_1 \otimes H) \circ \delta_H \circ \Pi_H^R \\ &= p_{A \otimes H} \circ \nabla_{A \otimes H} \circ (\eta_A \otimes \Pi_H^R) \\ &= f \circ \Pi_H^R \\ &= (A \times H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H, A \times H} \otimes H) \circ (H \otimes (\rho_{A \times_\sigma H} \circ \eta_{A \times_\sigma H})). \end{aligned}$$

In the previous calculations, the first equality follows by the normalized condition for the product $\mu_{A \otimes_\sigma H}$; the second one uses that (A, φ_A) is a weak left H -module, the coassociativity of δ_H and the naturality of c ; the third one relies on the coassociativity of δ_H , the naturality of c and because σ is in $\text{Reg}_{\varphi_A}(H \otimes H, A)$; the fourth one is a consequence of the coassociativity of δ_H and the naturality of c . The fifth uses that $\sigma \in \text{Reg}_{\varphi_A}(H \otimes H, A)$. The sixth follows by the definition of Π_H^R and that H is a weak Hopf algebra; the seventh one by (81), the eighth is a consequence of the equality $p_{A \otimes H} \circ \nabla_{A \otimes H} = p_{A \otimes H}$; finally, in the last one we use that f is a total integral.

In order to give condition (c2), we will show the equality

$$((\sigma \circ (H \otimes \mu_H)) \wedge (\sigma^{-1} \circ (\mu_H \otimes H))) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) \circ \delta_H = u_1. \quad (115)$$

Indeed: Using the anticomultiplicativity of the antipode, the coassociativity of δ_H , the naturality of c , the equalities (15) and (16) as well as $\sigma \in \text{Reg}_{\varphi_A}(H \otimes H, A)$ and (72), we get (115), because

$$((\sigma \circ (H \otimes \mu_H)) \wedge (\sigma^{-1} \circ (\mu_H \otimes H))) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) \circ \delta_H$$

$$\begin{aligned}
&= \mu_A \circ (\sigma \otimes \sigma^{-1}) \circ (H \otimes c_{H,H} \otimes H) \circ (((H \otimes \Pi_H^L) \circ \delta_H) \otimes ((\Pi_H^R \otimes H) \circ \delta_H)) \circ \delta_H \\
&= ((\sigma \circ (\mu_H \otimes H)) \wedge (\sigma^{-1} \circ (H \otimes \mu_H))) \circ (\eta_H \otimes H \otimes \eta_H) \\
&= \mu_A \circ (\sigma^{-1} \otimes \varphi_A) \circ (H \otimes c_{H,H} \otimes (\sigma \circ (H \otimes \eta_H))) \circ ((\delta_H \circ \eta_H) \otimes \delta_H) \\
&= (\sigma^{-1} \wedge u_2) \circ (\eta_H \otimes H) \\
&= \sigma^{-1} \circ (\eta_H \otimes H) \\
&= u_1.
\end{aligned}$$

Now we prove (c2):

$$\begin{aligned}
&f \wedge f^{-1} \\
&= p_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes ((\sigma^{-1} \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \lambda_H) \otimes H) \circ \delta_H)) \circ \delta_H \\
&= p_{A \otimes H} \circ ((\mu_A \circ ((\varphi_A \circ (H \otimes \sigma^{-1})) \otimes \sigma)) \circ (H \otimes H \otimes c_{H,H} \otimes H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \\
&\quad \circ (\delta_H \otimes \delta_H \otimes H)) \otimes H) \circ (H \otimes H \otimes c_{H,H}) \circ (H \otimes ((H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \lambda_H)))) \otimes H) \\
&\quad \circ (\delta_H \otimes \delta_H) \circ \delta_H \\
&= p_{A \otimes H} \circ ((\mu_A \circ (\sigma \otimes \sigma^{-1})) \circ (H \otimes \mu_H \otimes \mu_H \otimes H) \circ \delta_{H \otimes H \otimes H}) \otimes H) \circ (H \otimes H \otimes c_{H,H}) \\
&\quad \circ (H \otimes ((H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \lambda_H)))) \otimes H) \circ (\delta_H \otimes \delta_H) \circ \delta_H \\
&= p_{A \otimes H} \circ ((\mu_A \circ (\sigma \otimes \sigma^{-1})) \circ (H \otimes \mu_H \otimes \mu_H \otimes H) \circ \delta_{H \otimes H \otimes H}) \otimes H) \circ (H \otimes H \otimes c_{H,H}) \\
&\quad \circ (H \otimes ((H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes ((\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H)))) \otimes H) \circ (\delta_H \otimes \delta_H) \circ \delta_H \\
&= p_{A \otimes H} \circ ((\mu_A \circ (\sigma \otimes \sigma^{-1})) \circ (H \otimes \mu_H \otimes \mu_H \otimes H) \circ \delta_{H \otimes H \otimes H}) \otimes H) \circ (H \otimes H \otimes c_{H,H}) \\
&\quad \circ (H \otimes (c_{H,H} \circ (\Pi_H^L \otimes \lambda_H))) \otimes H) \circ (\delta_H \otimes \delta_H) \circ \delta_H \\
&= p_{A \otimes H} \circ ((\mu_A \circ (\sigma \otimes \sigma^{-1})) \circ (H \otimes \mu_H \otimes \mu_H \otimes H) \circ \delta_{H \otimes H \otimes H}) \otimes H) \\
&\quad \circ (H \otimes (((\lambda_H \otimes H) \circ \delta_H) \otimes \Pi_H^L) \circ c_{H,H} \circ \delta_H)) \circ \delta_H \\
&= p_{A \otimes H} \circ (((\sigma \circ (H \otimes \mu_H)) \wedge (\sigma^{-1} \circ (\mu_H \otimes H))) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) \circ \delta_H) \otimes \Pi_H^L) \circ \delta_H \\
&= p_{A \otimes H} \circ (u_1 \otimes \Pi_H^L) \circ \delta_H \\
&= p_{A \otimes H} \circ (A \otimes \Pi_H^L) \circ \nabla_{A \otimes H} \circ (\eta_A \otimes H) \\
&= p_{A \otimes H} \circ \nabla_{A \otimes H} \circ (\eta_A \otimes \bar{\Pi}_H^L) \\
&= f \circ \bar{\Pi}_H^L \\
&= (A \times H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_{A \times_\sigma H} \circ \eta_{A \times_\sigma H}) \otimes H).
\end{aligned}$$

The first equality follows by (79), the second one by the naturality of c and the coassociativity of δ_H , the third one by (71), the fourth one by the antimultiplicative property of λ_H , the fifth one by the naturality of c and the definition of Π_H^L . In the sixth one we use the naturality of c and the seventh follows by the cocommutativity of H and the coassociativity of δ_H . The eight one is a consequence of

(115) and the ninth of (81). In the tenth equality we use that if H is cocommutative $\Pi_H^L = \overline{\Pi}_H^L$ and $(A \otimes \overline{\Pi}_H^L) \circ \nabla_{A \otimes H} = \nabla_{A \otimes H} \circ (A \otimes \overline{\Pi}_H^L)$. The eleventh one relies on the properties of $\nabla_{A \otimes H}$ and finally the last one follows because f is a total integral.

To finish the proof we only need to show that $f^{-1} \wedge f \wedge f^{-1} = f^{-1}$. First of all, using that H is cocommutative it is not difficult to see that $(f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H = \rho_{A \times_\sigma H} \circ f^{-1}$. Then by this equality, the fact that $\lambda_H \circ \lambda_H = id_H$ (which follows because H is cocommutative) and (9),

$$\begin{aligned}
& f^{-1} \wedge f \wedge f^{-1} \\
&= \mu_{A \times_\sigma H} \circ (f^{-1} \otimes ((f \wedge f^{-1}) \circ \lambda_H \circ \lambda_H)) \circ \delta_H \\
&= \mu_{A \times_\sigma H} \circ (f^{-1} \otimes (f \circ \overline{\Pi}_H^L \circ \lambda_H \circ \lambda_H)) \circ c_{H,H} \circ \delta_H \\
&= \mu_{A \times_\sigma H} \circ (A \times H \otimes (f \circ \Pi_H^R)) \circ \rho_{A \times H} \circ f^{-1} \\
&= \mu_{A \times_\sigma H} \circ (A \times H \otimes (f^{-1} \wedge f)) \circ \rho_{A \times H} \circ f^{-1} \\
&= f^{-1},
\end{aligned}$$

and we conclude the proof. \square

Proposition 3.13. *Let H be a cocommutative weak Hopf algebra and let A be an algebra. If (φ_A, α) and (ϕ_A, β) are two equivalent crossed systems, so are the associated H -cleft extensions $A \hookrightarrow A \times_\alpha H$ and $A \hookrightarrow A \times_\beta H$.*

Proof:

We will begin showing that this correspondence is well defined: Let h be the morphism in $Reg_{\varphi_A}(H, A) \cap Reg_{\phi_A}(H, A)$ satisfying conditions (73) and (74). We denote by $A \hookrightarrow A \times_\alpha H$ and $A \hookrightarrow A \times_\beta H$ the H -cleft extensions defined by (φ_A, α) and (ϕ_A, β) , respectively. We will show that

$$T = p_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes h \otimes H) \circ (A \otimes \delta_H) \circ i_{A \otimes H} \quad (116)$$

is a morphism of H -comodule algebras such that $T \circ i_{A \times_\alpha H} = i_{A \times_\beta H}$. Firstly of all, note that, by (107) the idempotent morphisms defined by the two crossed systems coincide. We denote it by $\nabla_{A \otimes H}$. Moreover,

$$\begin{aligned}
& T \circ \eta_{A \times_\alpha H} \\
&= p_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes h \otimes H) \circ (A \otimes \delta_H) \circ \nabla_{A \otimes H} \circ (\eta_A \otimes \eta_H) \\
&= p_{A \otimes H} \circ ((u_1 \wedge h) \otimes H) \circ \delta_H \circ \eta_H \\
&= p_{A \otimes H} \circ (h \otimes H) \circ \delta_H \circ \eta_H \\
&= p_{A \otimes H} \circ ((h \circ \overline{\Pi}_H^L) \otimes H) \circ \delta_H \circ \eta_H \\
&= p_{A \otimes H} \circ ((\varphi_A \circ (H \otimes \eta_A)) \otimes H) \circ \delta_H \circ \eta_H \\
&= \eta_{A \times_\beta H}.
\end{aligned}$$

In the foregoing calculations, the first and the last equalities follow by the definition of $\nabla_{A \otimes H}$; in the second one we use that $\nabla_{A \otimes H}$ is a morphism of left A -modules and right H -comodules; the third that h is in $Reg_{\varphi_A}(H, A)$. Finally, the fourth equality follows by (17) and the fifth one by (57).

To prove the multiplicative condition for T we need to fix a new notation and get two auxiliary identities. First note that, by 3.7, the crossed systems (φ_A, α) and (ϕ_A, β) define two quadruples

$$\begin{aligned}
& (A, H, \psi_{H,\alpha}^A = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A), \sigma_{H,\alpha}^A = (\alpha \otimes \mu_H) \circ \delta_{H \otimes H}) \\
& (A, H, \psi_{H,\beta}^A = (\phi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A), \sigma_{H,\beta}^A = (\beta \otimes \mu_H) \circ \delta_{H \otimes H})
\end{aligned}$$

that induce the corresponding weak crossed products. On the other hand, the following equalities hold:

$$\mu_A \circ (A \otimes h) \circ \sigma_{H,\alpha}^A = \mu_A \circ ((\mu_A \circ (h \otimes \phi_A) \circ (\delta_H \otimes h)) \otimes \beta) \circ \delta_{H \otimes H}, \quad (117)$$

$$(\nabla_{A \otimes H} \otimes H) \circ ((\mu_A \circ (h \otimes \phi_A) \circ (\delta_H \otimes A)) \otimes \delta_H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) = (\mu_A \otimes \delta_H) \circ (h \otimes \psi_{H,\beta}^A) \circ (\delta_H \otimes A). \quad (118)$$

The proof for (117) is as follows:

$$\begin{aligned} & \mu_A \circ (A \otimes h) \circ \sigma_{H,\alpha}^A \\ &= \mu_A \circ ((\mu_A \circ (\mu_A \circ h^{-1}) \circ ((\mu_A \circ (h \otimes \phi_A) \circ (\delta_H \otimes h)) \otimes \sigma_{H,\beta}^A)) \otimes (h \circ \mu_H)) \circ (\delta_{H \otimes H} \otimes H \otimes H) \circ \delta_{H \otimes H} \\ &= \mu_A \circ ((\mu_A \circ (h \otimes \phi_A) \circ (\delta_H \otimes h)) \otimes (\mu_A \circ (\beta \otimes (h^{-1} \wedge h)))) \circ (\delta_{H \otimes H} \otimes \mu_H) \circ \delta_{H \otimes H} \\ &= \mu_A \circ ((\mu_A \circ (h \otimes \phi_A) \circ (\delta_H \otimes h)) \otimes (\beta \wedge (\phi_A \circ (\mu_H \otimes \eta_A)))) \circ \delta_{H \otimes H} \\ &= \mu_A \circ ((\mu_A \circ (h \otimes \phi_A) \circ (\delta_H \otimes h)) \otimes \beta) \circ \delta_{H \otimes H} \end{aligned}$$

where the first equality follows by the equivalence between (φ_A, α) and (ϕ_A, β) , the second one by the coassociativity of $\delta_{H \otimes H}$ and (a1) of the definition of weak Hopf algebra, the third one by the coassociativity of $\delta_{H \otimes H}$ and because $h \in \text{Reg}_{\varphi_A}(H, A)$ and finally, in the last one we use that $\beta \in \text{Reg}_{\varphi_A}(H \otimes H, A)$.

Taking into account that $\nabla_{A \otimes H}$ is a morphism of left A -modules and right H -comodules, the coassociativity of δ_H and (80), we obtain (118) because:

$$\begin{aligned} & (\nabla_{A \otimes H} \otimes H) \circ ((\mu_A \circ (h \otimes \phi_A) \circ (\delta_H \otimes A)) \otimes \delta_H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) \\ &= (A \otimes \delta_H) \circ \nabla_{A \otimes H} \circ (\mu_A \otimes H) \circ (h \otimes \psi_{H,\beta}^A) \circ (\delta_H \otimes A) \\ &= (\mu_A \otimes \delta_H) \circ (h \otimes \psi_{H,\beta}^A) \circ (\delta_H \otimes A). \end{aligned}$$

We are now in position to show that T is a multiplicative morphism.

$$\begin{aligned} & T \circ \mu_{A \times_\alpha H} \\ &= p_{A \otimes H} \circ ((\mu_A \circ (A \otimes h)) \otimes H) \circ (A \otimes \delta_H) \circ \nabla_{A \otimes H} \circ (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_{H,\alpha}^A) \circ (A \otimes \psi_{H,\alpha}^A \otimes H) \\ & \quad \circ (i_{A \otimes H} \otimes i_{A \otimes H}) \\ &= p_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes (\mu_A \circ (A \otimes h) \circ \sigma_{H,\alpha}^A) \otimes \mu_H) \circ (\mu_A \otimes \delta_{H \otimes H}) \circ (A \otimes \psi_{H,\alpha}^A \otimes H) \circ (i_{A \otimes H} \otimes i_{A \otimes H}) \\ &= p_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \mu_A \otimes H) \\ & \quad \circ (A \otimes ((\mu_A \circ ((\mu_A \circ (h \otimes A)) \otimes A) \circ (H \otimes \phi_A \otimes h^{-1}) \circ (\delta_H \otimes c_{H,A}) \circ (\delta_H \otimes A))) \\ & \quad \otimes (\mu_A \circ ((\mu_A \circ (h \otimes \phi_A) \circ (\delta_H \otimes h)) \otimes \beta)) \circ \delta_{H \otimes H}) \otimes \mu_H) \circ (A \otimes H \otimes A \otimes \delta_{H \otimes H}) \\ & \quad \circ (A \otimes H \otimes c_{H,A} \otimes H) \circ (A \otimes \delta_H \otimes A \otimes H) \circ (i_{A \otimes H} \otimes i_{A \otimes H}) \\ &= p_{A \otimes H} \circ (\mu_A \otimes H) \circ (\mu_A \otimes A \otimes H) \circ ((\mu_A \circ (A \otimes (\mu_A \circ (h \otimes \phi_A) \circ (\delta_H \otimes A)))) \\ & \quad \otimes (\mu_A \circ ((h^{-1} \wedge h) \otimes \phi_A) \circ (\delta_H \otimes h)) \otimes \sigma_{H,\beta}^A) \circ (A \otimes H \otimes A \otimes \delta_{H \otimes H}) \circ (A \otimes H \otimes c_{H,A} \otimes H) \\ & \quad \circ (A \otimes \delta_H \otimes A \otimes H) \circ (i_{A \otimes H} \otimes i_{A \otimes H}) \\ &= p_{A \otimes H} \circ (\mu_A \otimes H) \circ (\mu_A \otimes A \otimes H) \circ (A \otimes (\phi_A \circ (H \otimes h) \otimes \sigma_{H,\beta}^A)) \circ (\nabla_{A \otimes H} \otimes H \otimes H \otimes H) \\ & \quad \circ (\mu_A \circ (A \otimes (\mu_A \circ (h \otimes \phi_A))) \otimes \delta_{H \otimes H}) \circ (A \otimes \delta_H \otimes c_{H,A} \otimes H) \circ (A \otimes \delta_H \otimes A \otimes H) \circ (i_{A \otimes H} \otimes i_{A \otimes H}) \\ &= p_{A \otimes H} \circ (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_{H,\beta}^A) \circ (\mu_A \otimes \phi_A \otimes H \otimes H) \circ (A \otimes A \otimes H \otimes c_{H,A} \otimes H) \end{aligned}$$

$$\begin{aligned}
& \circ (A \otimes ((\mu_A \otimes \delta_H) \circ (h \otimes \psi_{H,\beta}^A) \circ (\delta_H \otimes A)) \otimes ((h \otimes H) \circ \delta_H)) \circ \circ (i_{A \otimes H} \otimes i_{A \otimes H}) \\
& = p_{A \otimes H} \circ (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_{H,\beta}^A) \circ (\mu_A \otimes ((\mu_A \otimes H) \circ (A \otimes \psi_{H,\beta}^A) \circ (\psi_{H,\beta}^A \otimes A)) \otimes H) \\
& \quad \circ (A \otimes ((h \otimes H) \circ \delta_H) \otimes A \otimes ((h \otimes H) \circ \delta_H)) \circ (i_{A \otimes H} \otimes i_{A \otimes H}) \\
& = p_{A \otimes H} \circ (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_{H,\beta}^A) \circ (A \otimes \psi_{H,\beta}^A) \otimes H) \\
& \quad \circ (((\mu_A \circ (A \otimes h)) \otimes H) \circ (A \otimes \delta_H)) \otimes (((\mu_A \circ (A \otimes h)) \otimes H) \circ (A \otimes \delta_H)) \circ (i_{A \otimes H} \otimes i_{A \otimes H}) \\
& = \mu_{A \times_\beta H} \circ (T \otimes T).
\end{aligned}$$

In the previous calculations, the first equality follows by the definition, the second one because $\nabla_{A \otimes H}$ is a morphism of left A -modules and (91), (89) hold, the third one is a consequence of the equivalence between (φ_A, α) and (ϕ_A, β) and (117), the fourth one follows by the associativity of μ_A , the coassociativity of δ_H and the naturality of c , the fifth one by the associativity of μ_A , the sixth one by (118), the seventh one by the definition of $\psi_{H,\beta}^A$, the eight one by (77) and, finally, the ninth one by the normalized condition for $\mu_{A \otimes_\beta H}$.

Using that

$$i_{A \otimes H} \circ T = (\mu_A \otimes H) \circ (A \otimes h \otimes H) \circ (A \otimes \delta_H) \circ i_{A \otimes H}, \quad (119)$$

it is not difficult to see that T is a morphism of right H -comodules. Moreover, by associativity of μ_A , the coassociativity of δ_H , (17) and (57), we have

$$\begin{aligned}
& T \circ i_{A \times_\alpha H} \\
& = p_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes h \otimes H) \circ (A \otimes \delta_H) \circ \nabla_{A \otimes H} \circ (A \otimes \eta_H) \\
& = p_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes ((u_1 \wedge h) \otimes H) \circ (A \otimes \delta_H) \circ (A \otimes \eta_H) \\
& = p_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes (h \otimes H)) \circ (A \otimes \delta_H) \circ (A \otimes \eta_H) \\
& = p_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes (h \circ \overline{\Pi}_H^L) \otimes H) \circ (A \otimes \delta_H) \circ (A \otimes \eta_H) \\
& = p_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes u_1 \otimes H) \circ (A \otimes \delta_H) \circ (A \otimes \eta_H) \\
& = p_{A \otimes H} \circ \nabla_{A \otimes H} \circ (A \otimes \eta_H) \\
& = i_{A \times_\beta H}.
\end{aligned}$$

Therefore, the associated H -cleft extensions $A \hookrightarrow A \times_\alpha H$ and $A \hookrightarrow A \times_\beta H$ are equivalent. \square

Remark 3.14. In the conditions of the previous result, T is an isomorphism with inverse

$$T^{-1} = p_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes h^{-1} \otimes H) \circ (A \otimes \delta_H) \circ i_{A \otimes H}$$

because

$$\begin{aligned}
& T^{-1} \circ T \\
& = p_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes (h \wedge h^{-1}) \otimes H) \circ (A \otimes \delta_H) \circ i_{A \otimes H} \\
& = p_{A \otimes H} \circ (\mu_A \otimes H) \circ (A \otimes u_1 \otimes H) \circ (A \otimes \delta_H) \circ i_{A \otimes H} \\
& = p_{A \otimes H} \circ \nabla_{A \otimes H} \circ i_{A \otimes H} \\
& = id_{A \times_\alpha H}.
\end{aligned}$$

Proposition 3.15. *Let H be a cocommutative weak Hopf algebra. If $A_H \hookrightarrow A$ is an H -cleft extension, the morphism*

$$\sigma_A := (\mu_A \circ (f \otimes f)) \wedge (f^{-1} \circ \mu_H) : H \otimes H \rightarrow A,$$

where $f : H \rightarrow A$ is a convolution invertible total integral, factors through the equalizer i_A . Moreover, if $\varphi_{A_H} : H \otimes A_H \rightarrow A_H$ is the weak left H -module structure defined in Proposition 2.18, the factorization of σ_A is a morphism in $\text{Reg}_{\varphi_{A_H}}(H \otimes H, A_H)$ satisfying the normal condition (g3) and with convolution inverse the factorization through the equalizer i_A of the morphism

$$\sigma_A^{-1} := (f \circ \mu_H) \wedge (\mu_A \circ c_{A,A} \circ (f^{-1} \otimes f^{-1})).$$

Proof:

If $A_H \hookrightarrow A$ is a H -cleft extension, by Corollary 2.11, we have that $A_H \hookrightarrow A$ is a weak H -cleft extension. Then, by Proposition 1.17 of [2], we obtain that σ_A factors through the equalizer i_A and, if σ_{A_H} is the factorization of σ_A , the equality

$$\sigma_{A_H} = p_A \circ \mu_A \circ (f \otimes f) \quad (120)$$

holds.

On the other hand, the morphism σ_A^{-1} factors through the equalizer i_A because

$$\rho_A \circ \sigma_A^{-1} = (A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ \sigma_A^{-1}.$$

Indeed:

$$\begin{aligned} & \rho_A \circ \sigma_A^{-1} \\ &= \mu_{A \otimes H} \circ (\rho_A \otimes (\rho_A \circ \mu_A)) \circ (A \otimes c_{A,A}) \circ ((f \circ \mu_H) \otimes f^{-1} \otimes f^{-1}) \circ \delta_{H \otimes H} \\ &= \mu_{A \otimes H} \circ (f \otimes H \otimes \mu_{A \otimes H}) \circ (\mu_H \otimes \mu_H \otimes (\rho_A \circ f^{-1}) \otimes (\rho_A \circ f^{-1})) \circ (\delta_{H \otimes H} \otimes c_{H,H}) \circ \delta_{H \otimes H} \\ &= (\mu_A \otimes \mu_H) \circ (\mu_A \otimes A \otimes \mu_H \otimes H) \circ (A \otimes A \otimes c_{H,A} \otimes \mu_H \otimes H) \circ (f \otimes c_{H,A} \otimes c_{H,A} \otimes H \otimes H) \\ & \quad \circ (\mu_H \otimes H \otimes c_{H,A} \otimes c_{H,A} \otimes H) \circ (H \otimes c_{H,H} \otimes H \otimes (\rho_A \circ f^{-1}) \otimes (\rho_A \circ f^{-1})) \circ (\delta_H \otimes H \otimes \delta_H \otimes H) \\ & \quad \circ (H \otimes H \otimes c_{H,H}) \circ \delta_{H \otimes H} \\ &= (\mu_A \otimes \mu_H) \circ (f \otimes \mu_{A \otimes H} \otimes H) \circ (H \otimes c_{H,A} \otimes c_{H,A} \otimes H) \\ & \quad \circ (\mu_H \otimes H \otimes [\Gamma_A^H \circ (H \otimes f^{-1}) \circ \delta_H] \otimes (\rho_A \circ f^{-1})) \circ (H \otimes c_{H,H} \otimes c_{H,H}) \circ (\delta_H \otimes H \otimes H \otimes H) \circ \delta_{H \otimes H} \\ &= (\mu_A \otimes \mu_H) \circ (f \otimes \mu_{A \otimes H} \otimes H) \circ (H \otimes c_{H,A} \otimes c_{H,A} \otimes H) \\ & \quad \circ (\mu_H \otimes H \otimes [(A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ f^{-1}] \otimes (\rho_A \circ f^{-1})) \circ (H \otimes c_{H,H} \otimes c_{H,H}) \circ (\delta_H \otimes H \otimes H \otimes H) \circ \delta_{H \otimes H} \\ &= (\mu_A \otimes \mu_H) \circ (A \otimes \mu_A \otimes H \otimes (\mu_H \circ (\overline{\Pi}_H^R \otimes H))) \circ (f \otimes A \otimes c_{H,A} \otimes H \otimes H) \\ & \quad \circ (\mu_H \otimes c_{H,A} \otimes c_{H,A} \otimes H) \circ (H \otimes H \otimes H \otimes (\rho_A \circ f^{-1}) \otimes (\rho_A \circ f^{-1})) \circ (H \otimes H \otimes H \otimes c_{H,H}) \\ & \quad \circ (H \otimes H \otimes \delta_H \otimes H) \circ \delta_{H \otimes H} \\ &= (\mu_A \otimes H) \circ (f \otimes [(A \otimes \mu_H) \circ (c_{H,A} \otimes ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes \delta_H))]) \circ (H \otimes \mu_A \otimes H \otimes H) \\ & \quad \circ (H \otimes A \otimes c_{H,A} \otimes H) \circ (H \otimes (\rho_A \circ f^{-1}) \otimes (\rho_A \circ f^{-1})) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)] \\ & \quad \circ (\mu_H \otimes H \otimes H) \circ \delta_{H \otimes H} \\ &= (\mu_A \otimes H) \circ (f \otimes [(A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \mu_A \otimes H) \circ (H \otimes f^{-1} \otimes (\rho_A \circ f^{-1})) \\ & \quad \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)]) \circ (\mu_H \otimes H \otimes H) \circ \delta_{H \otimes H} \end{aligned}$$

$$\begin{aligned}
&= (\mu_A \otimes H) \circ (f \otimes [((\mu_A \circ c_{A,A} \circ (f^{-1} \otimes f^{-1})) \otimes \Pi_H^L) \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \otimes \delta_H) \otimes H)]) \\
&\quad \circ (\mu_H \otimes H \otimes H) \circ \delta_{H \otimes H} \\
&= (\mu_A \otimes H) \circ (f \otimes [(A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \mu_A \otimes H) \circ (H \otimes f^{-1} \otimes (\rho_A \circ f^{-1})) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)]) \\
&\quad \circ (\mu_H \otimes H \otimes H) \circ \delta_{H \otimes H} \\
&= (A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ \sigma_A^{-1}.
\end{aligned}$$

In the last computations, the first equality follows by the definitions, the second one because A is a right H -comodule algebra and H a weak Hopf algebra, the third one by the coassociativity of δ_H , the fourth one by the associativity of μ_A and μ_H and the fifth one by Theorem 2.10. In the sixth equality we use the associativity of μ_H and the seventh one follows by (14). To prove the eight equality we use the definition of right H -comodule algebra and in the ninth one we apply the identity

$$\begin{aligned}
&((\mu_A \circ c_{A,A} \circ (f^{-1} \otimes f^{-1})) \otimes \Pi_H^L) \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \otimes \delta_H) \otimes H) \\
&= (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \mu_A \otimes H) \circ (H \otimes f^{-1} \otimes (\rho_A \circ f^{-1})) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)
\end{aligned} \tag{121}$$

obtained in Proposition 2.7. Finally, the tenth one is obtained by (8) and the idempotent character of $\overline{\Pi}_H^R$ and the eleventh one by repetition of the previous computations but in inverse order.

Let $\sigma_{A_H}^{-1}$ be the factorization of σ_A^{-1} . We will finish the proof showing that σ_{A_H} is a morphism in $\text{Reg}_{\varphi_{A_H}}(H \otimes H, A_H)$ with inverse $\sigma_{A_H}^{-1}$. First of all, note that $A_H \hookrightarrow A$ is an H -cleft extension and then, by Proposition 2.8, the morphism $f \wedge f^{-1}$ factors through the equalizer i_A . Now, using that H is a weak Hopf algebra, f an integral, A an H -comodule algebra, A_H a weak H -module algebra and the equality (40) we obtain

$$\begin{aligned}
&i_A \circ (\sigma_{A_H} \wedge \sigma_{A_H}^{-1}) \\
&= \mu_A \circ (\mu_A \otimes \mu_A) \circ (((\mu_A \circ (f \otimes f)) \otimes ((f^{-1} \wedge f) \circ \mu_H)) \otimes ((f^{-1} \otimes f^{-1}) \circ c_{H,H})) \circ (H \otimes H \otimes \delta_{H \otimes H}) \circ \delta_{H \otimes H} \\
&= \mu_A \circ (\mu_A \otimes \mu_A) \circ (A \otimes (f^{-1} \wedge f) \otimes ((f^{-1} \otimes f^{-1}) \circ c_{H,H})) \circ (\mu_{A \otimes H} \otimes H \otimes H) \\
&\quad \circ ((\rho_A \circ f) \otimes (\rho_A \circ f) \otimes H \otimes H) \circ \delta_{H \otimes H} \\
&= \mu_A \circ ((\mu_A \circ (A \otimes (f^{-1} \wedge f)) \circ \rho_A) \otimes \mu_A) \circ ((\mu_A \circ (f \otimes f)) \otimes ((f^{-1} \otimes f^{-1}) \circ c_{H,H})) \circ \delta_{H \otimes H} \\
&= q_A \circ \mu_A \circ (f \otimes (f \wedge f^{-1})) \\
&= q_A \circ \mu_A \circ (f \otimes (q_A \circ \mu_A \circ (f \otimes (i_A \circ \eta_{A_H})))) \\
&= i_A \circ \varphi_{A_H} \circ (H \otimes (\varphi_{A_H} \circ (H \otimes \eta_{A_H}))) \\
&= i_A \circ \varphi_{A_H} \circ (\mu_H \otimes \eta_{A_H}),
\end{aligned}$$

and then, using that i_A is a monomorphism, $\sigma_{A_H} \wedge \sigma_{A_H}^{-1} = \varphi_{A_H} \circ (\mu_H \otimes \eta_{A_H})$.

To prove $\sigma_{A_H}^{-1} \wedge \sigma_{A_H} = \varphi_{A_H} \circ (\mu_H \circ (H \otimes \eta_{A_H}))$ we need to check the equality

$$e_A \circ \mu_H \circ (\Pi_H^R \otimes H) = e_A \circ \mu_H \tag{122}$$

where e_A is the morphism defined in (28). Indeed, using the equality (12) and the definition of weak Hopf algebra we have:

$$\begin{aligned}
&e_A \circ \mu_H \circ (\Pi_H^R \otimes H) \\
&= (A \otimes [((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes (c_{H,H} \otimes \delta_H) \otimes H)]) \circ (c_{H,A} \otimes H \otimes H) \circ (H \otimes c_{H,A} \otimes H) \\
&\quad \circ (H \otimes H \otimes (\rho_A \circ \eta_A))
\end{aligned}$$

$$\begin{aligned}
&= (A \otimes [\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H)]) \circ (c_{H,A} \otimes H \otimes H) \circ (H \otimes c_{H,A} \otimes H) \circ (H \otimes H \otimes (\rho_A \circ \eta_A)) \\
&= e_A \circ \mu_H.
\end{aligned}$$

Then, $\sigma_{AH}^{-1} \wedge \sigma_{AH} = \varphi_{AH} \circ (\mu_H \otimes \eta_{AH})$, because by composing with the equalizer i_A we have

$$\begin{aligned}
&i_A \circ (\sigma_{AH}^{-1} \wedge \sigma_{AH}) \\
&= q_A \circ \mu_A \circ (A \otimes \mu_A) \circ (A \otimes \mu_A \otimes \mu_A) \circ (f \otimes f^{-1} \otimes f^{-1} \otimes f \otimes f) \circ (\mu_H \otimes c_{H,H} \otimes H \otimes H) \\
&\quad \circ (\delta_{H \otimes H} \otimes H \otimes H) \circ \delta_{H \otimes H} \\
&= q_A \circ \mu_A \circ (f \otimes \mu_A) \circ (\mu_H \otimes (\mu_A \circ (f^{-1} \otimes f^{-1}))) \circ (\mu_A \circ (f \otimes f)) \circ (H \otimes H \otimes c_{H,H} \otimes H \otimes H) \\
&\quad \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (H \otimes \delta_{H \otimes H} \otimes H) \circ (\delta_H \otimes \delta_H) \\
&= q_A \circ \mu_A \circ ((f \circ \mu_H) \otimes (\mu_A \circ (f^{-1} \otimes A))) \circ (H \otimes \delta_H \otimes [\mu_A \circ (e_A \otimes f)]) \circ \delta_{H \otimes H} \\
&= q_A \circ \mu_A \circ ((f \circ \mu_H) \otimes (\mu_A \circ (f^{-1} \otimes A))) \circ (H \otimes \delta_H \otimes [(A \otimes \varepsilon_H) \circ \Gamma_A^H \circ (H \otimes f)]) \circ \delta_{H \otimes H} \\
&= q_A \circ \mu_A \circ ((f \circ \mu_H) \otimes (\mu_A \circ (f^{-1} \otimes f))) \circ (\varepsilon_H \circ \mu_H) \circ (H \otimes \delta_H \otimes c_{H,H} \otimes H) \\
&\quad \circ (H \otimes H \otimes H \otimes \delta_H) \circ \delta_{H \otimes H} \\
&= q_A \circ \mu_A \circ ((f \circ \mu_H) \otimes e_A \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes H \otimes c_{H,H} \otimes H) \circ (H \otimes H \otimes H \otimes \delta_H) \circ \delta_{H \otimes H} \\
&= q_A \circ \mu_A \circ ((f \circ \mu_H) \otimes (e_A \circ \mu_H \circ (\Pi_H^R \otimes H))) \circ \delta_{H \otimes H} \\
&= q_A \circ (f \wedge e_A) \circ \mu_H \\
&= q_A \circ f \circ \mu_H. \\
&= (f \wedge f^{-1}) \circ \mu_H. \\
&= i_A \circ \varphi_{AH} \circ (\mu_H \otimes \eta_{AH}),
\end{aligned}$$

The first equality follows from (120) and (43), the second one relies on the coassociativity of δ_H and the associativity of μ_A , the third one by the naturality of the braiding, the fourth one follows from the properties of Γ_A^H and the fifth one by $(A \otimes \varepsilon_H) \circ \Gamma_A^H \circ (H \otimes f) = (f \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H)$. The naturality of the braiding yields the sixth one and the equality (12) implies the seventh one. The eighth one is a consequence of (122) and the fact that H is a weak Hopf algebra. Finally, in the ninth one we use that $f \wedge e_A = f$, the tenth one follows because f is a morphism of right H -comodules and in the eleventh one we use the definition of φ_{AH} .

To prove (f2) we compose with the equalizer i_A

$$\begin{aligned}
&i_A \circ (\sigma_{AH} \wedge \sigma_{AH}^{-1} \wedge \sigma_{AH}) \\
&= i_A \circ (\sigma_{AH} \wedge (\varphi_{AH} \circ (\mu_H \otimes \eta_{AH}))) \\
&= (\mu_A \circ (f \otimes f)) \wedge (f^{-1} \circ \mu_H) \wedge ((f \wedge f^{-1}) \circ \mu_H) \\
&= (\mu_A \circ (f \otimes f)) \wedge ((f^{-1} \wedge f \wedge f^{-1}) \circ \mu_H) \\
&= i_A \circ \sigma_{AH},
\end{aligned}$$

and then $\sigma_{AH} \wedge \sigma_{AH}^{-1} \wedge \sigma_{AH} = \sigma_{AH}$. In a similar way, using that $f \wedge f^{-1} \wedge f = f$ we get (f3).

To finish the proof, we only need to show that σ_{A_H} satisfies the normal condition. Indeed: by the usual arguments

$$i_A \circ \sigma_{A_H} \circ (\eta_H \otimes H) = q_A \circ \mu_A \circ (\eta_A \otimes f) = q_A \circ f = f \wedge f^{-1} = i_A \circ \varphi_{A_H} \circ (H \otimes \eta_{A_H})$$

and

$$i_A \circ \sigma_{A_H} \circ (H \otimes \eta_H) = q_A \circ f = f \wedge f^{-1} = i_A \circ \varphi_{A_H} \circ (H \otimes \eta_{A_H}).$$

Therefore, $\sigma_{A_H} \circ (\eta_H \otimes H) = \sigma_{A_H} \circ (H \otimes \eta_H) = \varphi_{A_H} \circ (H \otimes \eta_{A_H})$. □

Corollary 3.16. *In the conditions of Proposition 3.15, the following assertions are equivalent:*

- (i) $\sigma_{A_H} = \varphi_{A_H} \circ (\mu_H \otimes \eta_{A_H})$.
- (ii) $\mu_A \circ (f \otimes f) = f \circ \mu_H$.

Proof:

(i) \Rightarrow (ii). Composing with i_A , and using the definition of σ_A

$$\begin{aligned} & (i_A \circ \sigma_{A_H}) \wedge (f \circ \mu_H) \\ &= (\mu_A \circ (f \otimes f)) \wedge (f^{-1} \circ \mu_H) \wedge (f \circ \mu_H) \\ &= (\mu_A \circ (f \otimes f)) \wedge u_2 \\ &= \mu_A \circ (A \otimes e_A) \circ \rho_A \circ \mu_A \circ (f \otimes f) \\ &= \mu_A \circ (f \otimes f) \end{aligned}$$

On the other hand, by the hypothesis, $(i_A \circ \sigma_{A_H}) \wedge (f \circ \mu_H) = u_2 \wedge (f \circ \mu_H) = (u_1 \wedge f) \circ \mu_H = f \circ \mu_H$ and we obtain (ii).

The converse is an easy consequence of (120). Indeed:

$$\begin{aligned} & i_A \circ \sigma_{A_H} \\ &= q_A \circ \mu_A \circ (f \otimes f) \\ &= (f \wedge f^{-1}) \circ \mu_H \\ &= u_2 \\ &= i_A \circ \varphi_{A_H} \circ (\mu_H \otimes \eta_{A_H}) \end{aligned}$$

and then $\sigma_{A_H} = \varphi_{A_H} \circ (\mu_H \otimes \eta_{A_H})$. □

In the next Theorem we prove that each H -cleft extension determines an unique equivalence class of crossed systems for H over A . First we need a fundamental result in the study of H -cleft extensions that generalizes Theorem 11 of [18].

Theorem 3.17. *Let H be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an H -cleft extension with f an associated convolution invertible total integral. Then, the pair $(\varphi_{A_H}, \sigma_{A_H})$ is a crossed system for H over A_H , where φ_{A_H} is the weak H -module structure defined in Proposition 2.18 and σ_{A_H} the morphism obtained in Proposition 3.15. Moreover, the H -cleft extensions $A_H \hookrightarrow A$ and $A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H$ are equivalent.*

Proof:

First note that in this case

$$\psi_H^{A_H} = (p_A \otimes H) \circ \rho_A \circ \mu_A \circ (f \otimes i_A)$$

and

$$\sigma_H^{A_H} = (p_A \otimes H) \circ \rho_A \circ \mu_A \circ (f \otimes f)$$

and then, by Proposition 3.13 of [4], we have that the quadruple $(A_H, H, \psi_H^{A_H}, \sigma_H^{A_H})$ satisfies the twisted and the cocycle conditions. Moreover, the normal condition for σ_{A_H} implies that there exists a preunit. Therefore, by the theory exposed in 3.7, we obtain that $(\varphi_{A_H}, \sigma_{A_H})$ is a crossed system for H over A_H . Moreover, by Lemma 3.11 of [4], we obtain that

$$\nabla_{A_H \otimes H} = (p_A \otimes H) \circ \rho_A \circ \mu_A \circ (i_A \otimes f). \quad (123)$$

By Proposition 3.12, we known that $A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H$ is an H -cleft extension. Also, by 3.10 of [4], there exists a right H -comodule algebra isomorphism

$$T = p_{A_H \otimes H} \circ (p_A \otimes H) \circ \rho_A : A \rightarrow A_H \times_{\sigma_{A_H}} H$$

such that

$$T^{-1} = \mu_A \circ (i_A \otimes f) \circ i_{A_H \otimes H}$$

and

$$T^{-1} \circ i_{A_H \times_{\sigma_{A_H}} H} = \mu_A \circ (i_A \otimes f) \circ \nabla_{A_H \otimes H} \circ (A_H \otimes \eta_H) = \mu_A \circ (i_A \otimes (f \circ \eta_H)) = i_A.$$

Therefore, we obtain that $A_H \hookrightarrow A$ and $A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H$ are equivalent. \square

Proposition 3.18. *Let H be a cocommutative weak Hopf algebra and let (φ_A, σ) be a crossed system for H over A . Let $A \hookrightarrow A \times_{\sigma} H$ be the H -cleft extension constructed in Proposition 3.12. Then, if (ϕ_A, τ) is the crossed system associated to the H -cleft extension $A \hookrightarrow A \times_{\sigma} H$, we have that $(\phi_A, \tau) = (\varphi_A, \sigma)$.*

Proof:

By Proposition 3.11 and Theorem 3.17, the convolution invertible total integral $f = p_{A \otimes H} \circ (\eta_A \otimes H)$ determines a crossed system (ϕ_A, τ) , where ϕ_A and τ are defined by

$$\phi_A = p_{A \times_{\sigma} H} \circ \mu_{A \times_{\sigma} H} \circ (f \otimes i_{A \times_{\sigma} H}) \quad (124)$$

and

$$\tau = p_{A \times_{\sigma} H} \circ \mu_{A \times_{\sigma} H} \circ (f \otimes f), \quad (125)$$

where $p_{A \times_{\sigma} H}$ is the factorization through the equalizer $i_{A \times_{\sigma} H} = p_{A \otimes H} \circ (A \otimes \eta_H)$ of the morphism $q_{A \times_{\sigma} H} = \mu_{A \times_{\sigma} H} \circ (A \times_{\sigma} H \otimes f^{-1}) \circ \rho_{A \times_{\sigma} H}$.

We will show that the crossed systems (φ_A, σ) and (ϕ_A, τ) coincide. First of all we prove the following equality:

$$u_1 = \mu_A \circ ((\varphi_A \circ (H \otimes \sigma^{-1})) \otimes \sigma) \circ (H \otimes H \otimes ((c_{H,H} \otimes H) \circ (H \otimes c_{H,H}))) \circ ((\delta_{H \otimes H} \circ (H \otimes \lambda_H) \circ \delta_H) \otimes H) \circ \delta_H. \quad (126)$$

Indeed:

$$\begin{aligned} & \mu_A \circ ((\varphi_A \circ (H \otimes \sigma^{-1})) \otimes \sigma) \circ (H \otimes H \otimes ((c_{H,H} \otimes H) \circ (H \otimes c_{H,H}))) \circ ((\delta_{H \otimes H} \circ (H \otimes \lambda_H) \circ \delta_H) \otimes H) \circ \delta_H \\ &= \mu_A \circ ((\varphi_A \circ (H \otimes \sigma^{-1})) \otimes \sigma) \circ (H \otimes H \otimes c_{H,H} \otimes (\lambda_H \wedge \Pi_H^L)) \circ (H \otimes c_{H,H} \otimes c_{H,H}) \\ & \quad \circ (\delta_H \otimes \lambda_H \otimes H \otimes H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H) \circ (H \otimes \delta_H) \circ \delta_H \\ &= \mu_A \circ ((\varphi_A \circ (H \otimes \sigma^{-1})) \otimes (\sigma \circ (H \otimes \mu_H))) \circ (H \otimes H \otimes c_{H,H} \otimes H \otimes \Pi_H^L) \\ & \quad \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes (c_{H,H} \circ \delta_H \otimes \lambda_H) \otimes (c_{H,H} \circ \delta_H)) \circ (H \otimes \delta_H) \circ \delta_H \\ &= ((\varphi_A \circ (H \otimes \sigma^{-1})) \wedge (\sigma \circ (H \otimes (\mu_H \circ (H \otimes \Pi_H^L)))) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) \circ \delta_H \\ &= ((\sigma \circ (H \otimes \mu_H)) \wedge (\sigma^{-1} \circ (\mu_H \otimes H))) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) \circ \delta_H \\ &= \mu_A \circ (\sigma \otimes \sigma^{-1}) \circ (H \otimes (c_{H,H} \circ (\Pi_H^L \otimes \Pi_H^R)) \otimes H) \circ (\delta_H \otimes \delta_H) \circ \delta_H \\ &= \mu_A \circ ((\sigma \circ (H \otimes \Pi_H^R) \circ \delta_H) \otimes (\sigma^{-1} \circ (\Pi_H^L \otimes H) \circ \delta_H)) \circ \delta_H \end{aligned}$$

$$= u_1 \wedge u_1$$

$$= u_1$$

In the previous computations, the first, third and fifth equalities follow by the properties of the antipode λ_H and (5); in the second and sixth ones we use that H is cocommutative; the fourth follows by (g2); the seventh is a consequence of (66) and (61); finally, in the last equality we use that A is a weak H -module algebra.

Now we can obtain a simple expression for the morphism $q_{A \times_\sigma H}$:

$$q_{A \times_\sigma H} = p_{A \otimes H} \circ (A \otimes \Pi_H^L) \circ i_{A \otimes H}. \quad (127)$$

$$\begin{aligned} & q_{A \times_\sigma H} \\ &= \mu_{A \times_\sigma H} \circ (p_{A \otimes H} \otimes f^{-1}) \circ (A \otimes \delta_H) \circ i_{A \otimes H} \\ &= p_{A \otimes H} \circ (\mu_A \otimes H) \circ (\mu_A \otimes \sigma \otimes \mu_H) \circ (A \otimes \varphi_A \otimes \delta_{H \otimes H}) \circ (A \otimes H \otimes c_{H,A} \otimes H) \\ &\quad \circ (A \otimes \delta_H \otimes \sigma^{-1} \otimes H) \circ (A \otimes H \otimes H \otimes c_{H,H}) \circ (A \otimes H \otimes (\delta_H \circ \lambda_H) \otimes H) \circ (A \otimes H \otimes \delta_H) \circ (A \otimes \delta_H) \circ i_{A \otimes H} \\ &= p_{A \otimes H} \circ (\mu_A \otimes \Pi_H^L) \circ (A \otimes \mu_A \otimes H) \circ (A \otimes ((H \otimes \varphi_A) \circ \sigma^{-1}) \otimes \sigma \otimes H) \circ (A \otimes H \otimes H \otimes c_{H,H} \otimes H \otimes H) \\ &\quad \circ (A \otimes H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (A \otimes \delta_H \otimes (\delta_H \circ \lambda_H) \otimes H) \circ (A \otimes H \otimes (c_{H,H} \circ \delta_H)) \\ &\quad \circ (A \otimes \delta_H) \circ i_{A \otimes H} \\ &= p_{A \otimes H} \circ (\mu_A \otimes \Pi_H^L) \circ (A \otimes (\mu_A \circ ((\varphi_A \circ (H \otimes \sigma^{-1})) \otimes \sigma) \circ (H \otimes H \otimes c_{H,H} \otimes H) \\ &\quad \circ (H \otimes c_{H,H} \otimes c_{H,H}) \circ (\delta_H \otimes (\delta_H \circ \lambda_H) \otimes H) \circ (H \otimes \delta_H) \circ \delta_H)) \circ i_{A \otimes H} \\ &= p_{A \otimes H} \circ (\mu_A \otimes \Pi_H^L) \circ (A \otimes u_1 \otimes H) \circ (A \otimes \delta_H) \circ i_{A \otimes H} \\ &= p_{A \otimes H} \circ (A \otimes \Pi_H^L) \circ \nabla_{A \otimes H} \circ i_{A \otimes H} \\ &= p_{A \otimes H} \circ (A \otimes \Pi_H^L) \circ i_{A \otimes H}. \end{aligned}$$

In the foregoing calculations, the first equality follows by the H -comodule structure for $A \times_\sigma H$; in the second one we use that $\mu_{A \otimes_\sigma H} \circ (\nabla_{A \otimes H} \otimes \nabla_{A \otimes H}) = \mu_{A \otimes_\sigma H}$, the third relies on the antimultiplicativity of the antipode; the fourth by cocommutativity of H . The fifth one follows by (126), the seventh is a consequence of the definition of $\nabla_{A \otimes H}$; finally the last one uses that $\nabla_{A \otimes H} \circ i_{A \otimes H} = i_{A \otimes H}$.

On the other hand, $q_{A \times_\sigma H} = i_{A \times_\sigma H} \circ p_{A \times_\sigma H} = p_{A \otimes H} \circ (p_{A \times_\sigma H} \otimes \eta_H)$ and then

$$(A \otimes \varepsilon_H) \circ i_{A \otimes H} \circ q_{A \times_\sigma H} = (A \otimes \varepsilon_H) \circ \nabla_{A \otimes H} \circ (p_{A \times_\sigma H} \otimes \eta_H) = p_{A \times_\sigma H}$$

As a consequence,

$$p_{A \times_\sigma H} = (A \otimes \varepsilon_H) \circ i_{A \otimes H} \quad (128)$$

because

$$\begin{aligned} p_{A \times_\sigma H} &= (A \otimes \varepsilon_H) \circ \nabla_{A \otimes H} \circ (A \otimes \Pi_H^L) \circ i_{A \otimes H} = (\mu_A \otimes \varepsilon_H) \otimes (A \otimes (\varphi_A \circ (\Pi_H^L \otimes \eta_A)) \otimes H) \circ i_{A \otimes H} \\ &= (A \otimes \varepsilon_H) \circ \nabla_{A \otimes H} \circ i_{A \otimes H} = (A \otimes \varepsilon_H) \circ i_{A \otimes H}. \end{aligned}$$

Using this equality it is not difficult to see that $(\varphi_A, \sigma) = (\phi_A, \tau)$. Indeed:

$$\begin{aligned} & \phi_A \\ &= p_{A \times_\sigma H} \circ \mu_{A \times_\sigma H} \circ (f \otimes i_{A \times_\sigma H}) \end{aligned}$$

$$\begin{aligned}
&= (\mu_A \otimes \varepsilon_H) \circ (\varphi_A \otimes \sigma \otimes \mu_H) \circ (H \otimes A \otimes \delta_{H \otimes H}) \circ (H \otimes c_{H,A} \otimes H) \circ (\delta_H \otimes A \otimes \eta_H) \\
&= \mu_A \circ (\varphi_A \otimes (\sigma \circ (H \otimes \Pi_H^R) \circ \delta_H)) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) \\
&= \mu_A \circ (\varphi_A \otimes u_1) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) \\
&= \varphi_A,
\end{aligned}$$

and

$$\begin{aligned}
&\tau \\
&= p_{A \times_\sigma H} \circ \mu_{A \times_\sigma H} \circ (f \otimes f) \\
&= (A \otimes \varepsilon_H) \circ \nabla_{A \otimes H} \circ (\sigma \otimes \mu_H) \circ \delta_{H \otimes H} \\
&= \sigma \wedge u_2 \\
&= \sigma.
\end{aligned}$$

□

The following theorem is the weak version of Lemma 2.1 of [17].

Proposition 3.19. *Let H be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an H -cleft extension with f an associated convolution invertible total integral. Assume that $g : H \rightarrow A$ is another convolution invertible total integral with associated crossed system (ϕ_{A_H}, τ_{A_H}) . Then the crossed systems $(\varphi_{A_H}, \sigma_{A_H})$ and (ϕ_{A_H}, τ_{A_H}) are equivalent.*

Proof:

The morphism $\tilde{h} = f \wedge g^{-1}$ factors through the equalizer i_A . Indeed, by (32), the coassociativity of δ_H and the naturality of c , we have

$$\begin{aligned}
&\rho_A \circ \tilde{h} \\
&= (\rho_A \circ f) \wedge (\rho_A \circ g^{-1}) \\
&= ((f \otimes H) \circ \delta_H) \wedge ((g^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H) \\
&= ((\mu_A \circ (f \otimes g^{-1})) \otimes \Pi_H^L) \circ (H \otimes (c_{H,H} \circ \delta_H)) \circ \delta_H \\
&= ((\mu_A \circ (f \otimes g^{-1})) \otimes (\Pi_H^L \circ \Pi_H^L)) \circ (H \otimes (c_{H,H} \circ \delta_H)) \circ \delta_H \\
&= (A \otimes \Pi_H^L) \circ \rho_A \circ \tilde{h},
\end{aligned}$$

and then there exists a morphism $h : H \rightarrow A_H$ such that $\tilde{h} = i_A \circ h$. Note that, in the conditions of this theorem, $f \wedge f^{-1} = g \wedge g^{-1}$, and $f^{-1} \wedge f = g^{-1} \wedge g$. Then,

$$\varphi_{A_H} \circ (H \otimes \eta_{A_H}) = \phi_{A_H} \circ (H \otimes \eta_{A_H}) \quad (129)$$

because

$$i_A \circ \varphi_{A_H} \circ (H \otimes \eta_{A_H}) = q_A \circ f = f \wedge f^{-1}$$

and

$$i_A \circ \phi_{A_H} \circ (H \otimes \eta_{A_H}) = q'_A \circ g = g \wedge g^{-1}$$

where q_A is the morphism defined in Remark 2.12 and q'_A the analogous for g .

On the other hand, using that f and g are convolution invertible total integrals, we have

$$\begin{aligned}
&\tilde{h} \circ \eta_H \\
&= \mu_A \circ (A \otimes g^{-1}) \circ \rho_A \circ f \circ \eta_H
\end{aligned}$$

$$\begin{aligned}
&= \mu_A \circ (A \otimes g^{-1}) \circ \rho_A \circ g \circ \eta_H \\
&= (g \wedge g^{-1}) \circ \eta_H \\
&= \eta_A,
\end{aligned}$$

Therefore, taking into account that $\eta_A = i_A \circ \eta_{A_H}$, we obtain that $h \circ \eta_H = \eta_{A_H}$.

The morphism $\tilde{h}^{-1} = g \wedge f^{-1}$ admits a factorization through the equalizer i_A (the proof is similar to the one developed for \tilde{h}) and the factorization h^{-1} is the convolution inverse of h . As a consequence h is in $\text{Reg}_{\varphi_{A_H}}(H, A_H) \cap \text{Reg}_{\phi_{A_H}}(H, A_H)$. Indeed: First note that

$$\begin{aligned}
i_A \circ (h \wedge h^{-1}) &= \tilde{h} \wedge \tilde{h}^{-1} = f \wedge g^{-1} \wedge g \wedge f^{-1} = f \wedge f^{-1} \wedge f \wedge f^{-1} = \varphi_A \circ (H \otimes \eta_A) = i_A \circ \varphi_{A_H} \circ (H \otimes \eta_{A_H}) \\
\text{and, by (129), } h \wedge h^{-1} &= \varphi_{A_H} \circ (H \otimes \eta_{A_H}) = \phi_{A_H} \circ (H \otimes \eta_{A_H}). \text{ Similarly, } h^{-1} \wedge h = \varphi_{A_H} \circ (H \otimes \eta_{A_H}) = \\
&\phi_{A_H} \circ (H \otimes \eta_{A_H}). \text{ Moreover,}
\end{aligned}$$

$$i_A \circ (h \wedge h^{-1} \wedge h) = \tilde{h} \wedge \tilde{h}^{-1} \wedge \tilde{h} = \tilde{h} = i_A \circ h$$

and

$$i_A \circ (h^{-1} \wedge h \wedge h^{-1}) = \tilde{h}^{-1} \wedge \tilde{h} \wedge \tilde{h}^{-1} = \tilde{h}^{-1} = i_A \circ h^{-1}$$

Then $h \wedge h^{-1} \wedge h = h$ and $h^{-1} \wedge h \wedge h^{-1} = h^{-1}$.

The proof for (73) follows by the definition of h and ϕ_{A_H} . Indeed:

$$\begin{aligned}
&i_A \circ \mu_{A_H} \circ (\mu_{A_H} \otimes A_H) \circ (h \otimes \phi_{A_H} \otimes h^{-1}) \circ (\delta_H \otimes c_{H, A_H}) \circ (\delta_H \otimes A_H) \\
&= \mu_A \circ (\mu_A \otimes (f \wedge g^{-1} \wedge g)) \circ ((g^{-1} \wedge g \wedge f^{-1}) \otimes c_{H, A}) \circ (\delta_H \otimes i_A) \\
&= \mu_A \circ (\mu_A \otimes (f^{-1} \wedge f \wedge f^{-1})) \circ ((f \wedge f^{-1} \wedge f) \otimes c_{H, A}) \circ (\delta_H \otimes i_A) \\
&= i_A \circ \varphi_{A_H},
\end{aligned}$$

and then $\varphi_{A_H} = \mu_{A_H} \circ (\mu_{A_H} \otimes A_H) \circ (h \otimes \phi_{A_H} \otimes h^{-1}) \circ (\delta_H \otimes c_{H, A_H}) \circ (\delta_H \otimes A_H)$.

In order to get (74), we begin by showing the equality

$$\mu_A \circ \mu_{A \otimes A} \circ (f \otimes (g^{-1} \wedge g) \otimes (f \wedge g^{-1}) \otimes g) \circ (\delta_H \otimes \delta_H) = \mu_A \circ (f \otimes f), \quad (130)$$

which follows because f and g are convolution invertible integrals, A a right H -comodule algebra, (11) and the equality $f^{-1} \wedge f = g^{-1} \wedge g$. Indeed:

$$\begin{aligned}
&\mu_A \circ \mu_{A \otimes A} \circ (f \otimes (g^{-1} \wedge g) \otimes (f \wedge g^{-1}) \otimes g) \circ (\delta_H \otimes \delta_H) \\
&= (A \otimes \varepsilon_H) \circ \mu_{A \otimes H} \circ (\mu_A \otimes H \otimes ((\mu_A \otimes H) \circ (A \otimes c_{H, A}) \circ ((\rho_A \circ \eta_A) \otimes A))) \circ (A \otimes c_{H, A} \otimes A) \\
&\quad \circ ((\mu_A \circ (f \otimes f)) \otimes H \otimes ((g^{-1} \otimes g) \circ \delta_H)) \circ \delta_{H \otimes H} \\
&= (A \otimes \varepsilon_H) \circ \mu_{A \otimes H} \circ ((\mu_A \circ (f \otimes f)) \otimes H \otimes ((\mu_A \otimes \Pi_H^L) \circ (A \otimes \rho_A) \circ (g^{-1} \otimes g) \circ \delta_H)) \circ \delta_{H \otimes H} \\
&= (A \otimes \varepsilon_H) \circ \mu_{A \otimes H} \circ ((\mu_A \circ (f \otimes f)) \otimes H \otimes ((g^{-1} \wedge g) \otimes \Pi_H^L) \circ \delta_H) \circ \delta_{H \otimes H} \\
&= (A \otimes \varepsilon_H) \circ \mu_{A \otimes H} \circ (((f \otimes H) \circ \delta_H) \otimes (((f \wedge f^{-1} \wedge f) \otimes H) \circ \delta_H)) \\
&= (A \otimes \varepsilon_H) \circ \mu_{A \otimes H} \circ (((f \otimes H) \circ \delta_H) \otimes ((f \otimes H) \circ \delta_H)) \\
&= \mu_A \circ (f \otimes f).
\end{aligned}$$

Using this equality and similar arguments to the ones developed above we will finish the proof showing (74):

$$\begin{aligned}
&i_A \circ \mu_{A_H} \circ (\mu_{A_H} \otimes h^{-1}) \circ (\mu_{A_H} \otimes \tau_{A_H} \otimes \mu_H) \circ (h \otimes \phi_{A_H} \otimes \delta_{H \otimes H}) \circ (\delta_H \otimes h \otimes H \otimes H) \circ \delta_{H \otimes H} \\
&= \mu_A \circ ((\mu_A \circ (\mu_A \otimes g^{-1})) \circ ((f \wedge f^{-1} \wedge f) \otimes (f \wedge g^{-1}) \otimes H) \circ (H \otimes c_{H, H}) \circ (\delta_H \otimes H))
\end{aligned}$$

$$\begin{aligned}
& \otimes (\mu_A \circ (A \otimes (f^{-1} \wedge f \wedge f^{-1})) \circ \rho_A \circ \mu_A \circ (g \otimes g)) \circ \delta_{H \otimes H} \\
&= (\mu_A \circ \mu_{A \otimes A} \circ (f \otimes (g^{-1} \wedge g) \otimes (f \wedge g^{-1}) \otimes g) \circ (\delta_H \otimes \delta_H)) \wedge (f^{-1} \circ \mu_H) \\
&= (\mu_A \circ (f \otimes f)) \wedge (f^{-1} \circ \mu_H) \\
&= i_A \circ \sigma_{A_H}.
\end{aligned}$$

□

Corollary 3.20. *Let H be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$, $A_H \hookrightarrow B$ two equivalent H -cleft extensions with associated convolution invertible total integrals f and g respectively. Then the corresponding crossed systems $(\varphi_{A_H}, \sigma_{A_H})$ and (ϕ_{A_H}, τ_{A_H}) are equivalent.*

Proof:

If $A_H \hookrightarrow A$ and $A_H \hookrightarrow B$ are equivalent, there exists an isomorphism of right H -comodule algebras $T : A \rightarrow B$ such that $i_B = T \circ i_A$, and, as a consequence, $l = T \circ f$ is a convolution invertible total integral for $A_H \hookrightarrow B$ with inverse $l^{-1} = T \circ f^{-1}$. Therefore, by Proposition 3.19, the crossed system $(\psi_{A_H}, \omega_{A_H})$ associated to $A_H \hookrightarrow B$ for l is equivalent to (ϕ_{A_H}, τ_{A_H}) . Moreover, if p_A is the factorization through i_A of the morphism $q_A = \mu_A \circ (A \otimes f^{-1}) \circ \rho_A$ and p_B is the factorization through i_B of the morphism $q_B = \mu_A \circ (B \otimes l^{-1}) \circ \rho_B$, we have the following: By (48)

$$\psi_{A_H} = p_B \circ \mu_B \circ (l \otimes i_B)$$

and then

$$\psi_{A_H} = p_B \circ \mu_B \circ (T \otimes T) \circ (f \otimes i_A) = p_B \circ T \circ \mu_A \circ (f \otimes i_A) = p_A \circ \mu_A \circ (f \otimes i_A) = \varphi_{A_H}.$$

On the other hand, by (120)

$$\omega_{A_H} = p_B \circ \mu_B \circ (l \otimes l)$$

and, as a consequence,

$$\omega_{A_H} = p_B \circ \mu_B \circ (T \otimes T) \circ (f \otimes f) = p_B \circ T \circ \mu_A \circ (f \otimes f) = p_A \circ \mu_A \circ (f \otimes f) = \sigma_{A_H}.$$

Therefore $(\varphi_{A_H}, \sigma_{A_H})$ and (ϕ_{A_H}, τ_{A_H}) are equivalent. □

Theorem 3.21. *Let H be a cocommutative weak Hopf algebra. Two H -cleft extensions $A_H \hookrightarrow A$, $A_H \hookrightarrow B$ are equivalent if and only if so are their respective associated crossed systems.*

Proof:

The "if" part is a consequence of the previous corollary. Moreover, if $A_H \hookrightarrow A$, $A_H \hookrightarrow B$ are two weak H -cleft extensions with equivalent crossed systems $(\varphi_{A_H}, \sigma_{A_H})$ and (ϕ_{A_H}, τ_{A_H}) , by Proposition 3.13 we know that the associated H -cleft extensions $A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H$ and $A_H \hookrightarrow A_H \times_{\tau_{A_H}} H$ are equivalent. Therefore, by Theorem 3.17, we obtain

$$A_H \hookrightarrow A \approx A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H \approx A_H \hookrightarrow A_H \times_{\tau_{A_H}} H \approx A_H \hookrightarrow B$$

which proves the Theorem. □

Now we can give the main result of this section which is a generalization of Theorem 2.7 of [17].

Theorem 3.22. *Let H be a cocommutative weak Hopf algebra and (A, ρ_A) a right H -comodule algebra. There exists a bijective correspondence between the equivalence classes of H -cleft extensions $A_H \hookrightarrow B$ and the equivalence classes of crossed systems for H over A_H .*

Proof:

If $CS(H, A_H)$ denotes the set of equivalence classes of crossed systems of H over A_H and $Cleft(A_H)$ the set of equivalence classes of H -cleft extensions $A_H \hookrightarrow B$, by Proposition 3.13 and Corollary 3.20 we have two maps

$$F : CS(H, A_H) \rightarrow Cleft(A_H), \quad G : Cleft(A_H) \rightarrow CS(H, A_H)$$

defined by

$$F([\varphi_{A_H}, \sigma_{A_H}]) = [A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H]$$

and

$$G([A_H \hookrightarrow B]) = [(\phi_{A_H}, \tau_{A_H})].$$

The map G is the inverse of F , because, by Proposition 3.18, we have

$$(G \circ F)([\varphi_{A_H}, \sigma_{A_H}]) = G([A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H]) = [(\varphi_{A_H}, \sigma_{A_H})],$$

and by Theorem 3.17

$$(F \circ G)([A_H \hookrightarrow B]) = F([\phi_{A_H}, \tau_{A_H}]) = [A_H \hookrightarrow A_H \times_{\tau_{A_H}} H] = [A_H \hookrightarrow B].$$

□

4. CROSSED SYSTEMS AND COHOMOLOGY

In [6] we have developed a cohomology theory of algebras over weak Hopf algebras which generalizes the one given in [26] for Hopf algebras. The main result contained in [6] (see Theorem 3.11) asserts that if (A, φ_A) is a commutative left H -module algebra, there exists a bijection between the second cohomology group, denoted by $H_{\varphi_A}^2(H, A)$, and the equivalence classes of weak crossed products $A \otimes_{\alpha} H$ where $\alpha : H \otimes H \rightarrow A$ satisfies the 2-cocycle and the normal conditions. In this section, for a cocommutative weak Hopf algebra and an H -cleft extension $A_H \hookrightarrow A$, we will establish a bijection between the set of equivalence classes of crossed systems with a fixed weak H -module algebra structure and the second cohomology group $H_{\varphi_{Z(A_H)}}^2(H, Z(A_H))$, being $Z(A_H)$ the center of the subalgebra of coinvariants A_H . Our results generalizes to the weak Hopf algebra setting the ones proved by Doi for Hopf algebras in [17].

Proposition 4.1. *Let H be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an H -cleft extension. We denote by $(\varphi_{A_H}, \sigma_{A_H})$ the corresponding crossed system defined by the convolution invertible total integral $f : H \rightarrow A$. Then $(Z(A_H), \varphi_{Z(A_H)})$ is a left H -module algebra, where $\varphi_{Z(A_H)}$ is the factorization through the equalizer $i_{Z(A_H)}$ of the morphism $\varphi_{A_H} \circ (H \otimes i_{Z(A_H)})$.*

Proof:

We define $\psi_A : H \otimes A_H \rightarrow A$ as $\psi_A = \mu_A \circ (A \otimes (\mu_A \circ c_{A,A})) \circ (((f^{-1} \otimes f) \circ \delta_H) \otimes i_A)$. In a similar way to Proposition 2.18, it is not difficult to see that ψ_A factors through the equalizer i_A , and then there exists a morphism $\psi_{A_H} : H \otimes A_H \rightarrow A_H$ such that $i_A \circ \psi_{A_H} = \psi_A$. On the other hand, the following equalities hold:

$$\mu_A \circ (f^{-1} \otimes i_A) = \mu_A \circ (\psi_A \otimes f^{-1}) \circ (H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H). \quad (131)$$

$$\mu_A \circ c_{A,A} \circ (f \otimes i_A) = \mu_A \circ (\psi_A \otimes f) \circ (\delta_H \otimes A_H). \quad (132)$$

Indeed, using (e3), (e1) and that u_1 factors through the center of A (which follows because H is cocommutative and then (d4) and (d5) coincide),

$$\begin{aligned} & \mu_A \circ (f^{-1} \otimes i_A) \\ &= \mu_A \circ ((f^{-1} \wedge u_1) \otimes i_A) \\ &= \mu_A \circ (\mu_A \otimes A) \circ (f^{-1} \otimes i_A \otimes u_1) \circ (H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H) \\ &= \mu_A \circ (\mu_A \otimes A) \circ (f^{-1} \otimes i_A \otimes (f \wedge f^{-1})) \circ (H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H) \\ &= \mu_A \circ (\psi_A \otimes f^{-1}) \circ (H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H). \end{aligned}$$

The proof for (132) follows a similar pattern.

Now we can prove that $\varphi_{A_H} \circ (H \otimes i_{Z(A_H)})$ factors through the center of A_H . Indeed:

$$i_A \circ \mu_{A_H} \circ ((\varphi_{A_H} \circ (H \otimes i_{Z(A_H)})) \otimes A_H)$$

$$\begin{aligned}
&= \mu_A \circ ((\mu_A \circ (f \otimes i_A)) \otimes (\mu_A \circ (f^{-1} \otimes i_A))) \circ (H \otimes c_{H,A_H} \otimes A_H) \circ (\delta_H \otimes i_{Z(A_H)} \otimes A_H) \\
&= \mu_A \circ ((\mu_A \circ (f \otimes i_A)) \otimes (\mu_A \circ (\psi_A \otimes f^{-1}) \circ (H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H))) \circ (H \otimes c_{H,A_H} \otimes A_H) \\
&\quad \circ (\delta_H \otimes i_{Z(A_H)} \otimes A_H) \\
&= \mu_A \circ (\mu_A \otimes f^{-1}) \circ ((\mu_A \circ (f \otimes (i_A \circ i_{Z(A_H)}))) \otimes (i_A \circ \psi_{A_H}) \otimes H) \circ (H \otimes Z(A_H) \otimes H \otimes c_{H,A_H}) \\
&\quad \circ (H \otimes Z(A_H) \otimes \delta_H \otimes A_H) \circ (H \otimes c_{H,Z(A_H)} \otimes A_H) \circ (\delta_H \otimes Z(A_H) \otimes A_H) \\
&= \mu_A \circ (\mu_A \otimes A) \circ (f \otimes (\mu_A \circ c_{A,A} \circ ((i_A \circ i_{Z(A_H)}) \otimes (i_A \circ \psi_{A_H}))) \otimes f^{-1}) \circ (H \otimes Z(A_H) \otimes H \otimes c_{H,A_H}) \\
&\quad \circ (H \otimes Z(A_H) \otimes \delta_H \otimes A_H) \circ (H \otimes c_{H,Z(A_H)} \otimes A_H) \circ (\delta_H \otimes Z(A_H) \otimes A_H) \\
&= \mu_A \circ ((\mu_A \circ ((\mu_A \circ (f \wedge f^{-1} \otimes i_A)) \otimes f) \circ (H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H)) \otimes (\mu_A \circ ((i_A \circ i_{Z(A_H)}) \otimes f^{-1}))) \\
&\quad \circ (H \otimes c_{Z(A_H),A_H} \otimes H) \circ (H \otimes Z(A_H) \otimes c_{H,A_H}) \circ (H \otimes c_{H,Z(A_H)} \otimes A_H) \circ (\delta_H \otimes Z(A_H) \otimes A_H) \\
&= \mu_A \circ ((\mu_A \circ c_{A,A} \circ (f \otimes i_A)) \otimes (\mu_A \circ ((i_A \circ i_{Z(A_H)}) \otimes f^{-1}))) \circ (H \otimes c_{Z(A_H),A_H} \otimes H) \\
&\quad \circ (H \otimes Z(A_H) \otimes c_{H,A_H}) \circ (H \otimes c_{H,Z(A_H)} \otimes A_H) \circ (\delta_H \otimes Z(A_H) \otimes A_H) \\
&= i_A \circ \mu_{A_H} \circ c_{A_H,A_H} \circ ((\varphi_{A_H} \circ (H \otimes i_{Z(A_H)})) \otimes A_H)
\end{aligned}$$

In the foregoing computations, the first equality uses the definition of φ_A ; the second one relies on (131); the third and fifth ones are consequence of the definition of ψ_A , the fourth follows by the properties of the center of A_H ; finally in the last one we apply (132)

Then there exists a morphism $\varphi_{Z(A_H)} : H \otimes Z(A_H) \rightarrow Z(A_H)$ such that $i_{Z(A_H)} \circ \varphi_{Z(A_H)} = \varphi_{A_H} \circ (H \otimes i_{Z(A_H)})$. Trivially, $\varphi_{Z(A_H)}$ satisfies the conditions of Definition 2.16. In order to show that $(Z(A_H), \varphi_{Z(A_H)})$ is a left H -module algebra, we only need to prove the equality

$$\varphi_{Z(A_H)} \circ (H \otimes \varphi_{Z(A_H)}) = \varphi_{Z(A_H)} \circ (\mu_H \otimes Z(A_H)) \quad (133)$$

which follows composing with the equalizer $i_{Z(A_H)}$. Indeed:

$$\begin{aligned}
&i_{Z(A_H)} \circ \varphi_{Z(A_H)} \circ (H \otimes \varphi_{Z(A_H)}) \\
&= \varphi_{A_H} \circ (H \otimes \varphi_{A_H}) \circ (H \otimes H \otimes i_{Z(A_H)}) \\
&= \mu_{A_H} \circ (A_H \otimes (\mu_{A_H} \circ ((i_{Z(A_H)} \circ \varphi_{Z(A_H)}) \otimes A_H))) \circ (A_H \otimes H \otimes c_{A_H,Z(A_H)}) \circ (\sigma \otimes \mu_H \otimes \sigma^{-1} \otimes Z(A_H)) \\
&\quad \circ (H \otimes H \otimes \delta_{H \otimes H} \otimes Z(A_H)) \circ (\delta_{H \otimes H} \otimes Z(A_H)) \\
&= \mu_{A_H} \circ (\sigma \wedge \sigma^{-1} \otimes (i_{Z(A_H)} \circ \varphi_{Z(A_H)})) \circ (H \otimes H \otimes \mu_H \otimes Z(A_H)) \circ (\delta_{H \otimes H} \otimes Z(A_H)) \\
&= \mu_{A_H} \circ (u_2 \otimes (i_{Z(A_H)} \circ \varphi_{Z(A_H)})) \circ (H \otimes H \otimes \mu_H \otimes Z(A_H)) \circ (\delta_{H \otimes H} \otimes Z(A_H)) \\
&= \mu_{A_H} \circ (\varphi_{A_H} \otimes \varphi_{A_H}) \circ (H \otimes c_{H,A_H} \otimes A_H) \circ ((\delta_H \circ \mu_H) \otimes \eta_{A_H} \otimes i_{Z(A_H)}) \\
&= \varphi_{A_H} \circ (\mu_H \otimes i_{Z(A_H)})
\end{aligned}$$

In the above calculations, the first equality uses that φ_{A_H} factors through $i_{Z(A_H)}$; the second follows by (g1); the third one by the properties of the center of A_H ; the fourth because σ is a morphism in $\text{Reg}_{\varphi_{A_H}}(H, A_H)$; in the fifth equality we apply that H is a weak Hopf algebra; finally, the last one follows because (A_H, φ_{A_H}) is a weak H -module.

Taking into account that $i_{Z(A_H)}$ is a monomorphism, $\varphi_{Z(A_H)} \circ (H \otimes \varphi_{Z(A_H)}) = \varphi_{Z(A_H)} \circ (\mu_H \otimes Z(A_H))$ and we conclude the proof. \square

The following technical Lemma will be useful for the last Theorem of this paper.

Lemma 4.2. *Let H be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an H -cleft extension. We denote by φ_{A_H} the weak H -module algebra structure defined for A_H by a convolution invertible total integral $f : H \rightarrow A$, and by ψ_{A_H} the morphism defined in Proposition 4.1. Then the equality*

$$\varphi_{A_H} \circ (H \otimes \psi_{A_H}) \circ (\delta_H \otimes A_H) = \mu_{A_H} \circ ((\varphi_{A_H} \circ (H \otimes \eta_{A_H})) \otimes A_H) \quad (134)$$

holds.

Proof:

We compose the left part of the equality with the monomorphism i_A . Using that H is cocommutative and (131),

$$\begin{aligned} & i_A \circ \varphi_{A_H} \circ (H \otimes \psi_{A_H}) \circ (\delta_H \otimes A_H) \\ &= \mu_A \circ (\mu_A \otimes f \wedge f^{-1}) \circ (f \wedge f^{-1} \otimes c_{H,A}) \circ (\delta_H \otimes i_A) \\ &= \mu_A \circ (A \otimes \mu_A) \circ (f \otimes (\mu_A \circ (f^{-1} \otimes i_A))) \otimes f \wedge f^{-1} \circ (\delta_H \otimes c_{H,A}) \circ (\delta_H \otimes A_H) \\ &= \mu_A \circ (f \wedge f^{-1} \otimes i_A) \\ &= \mu_A \circ (u_1 \otimes i_A) \\ &= \mu_A \circ ((i_A \circ \varphi_{A_H} \circ (H \otimes \eta_{A_H})) \otimes i_A) \\ &= i_A \circ \mu_{A_H} \circ ((\varphi_{A_H} \circ (H \otimes \eta_{A_H})) \otimes A_H) \end{aligned}$$

\square

Now we will show the main result of this section.

Theorem 4.3. *Let H be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an H -cleft extension. We denote by $(\varphi_{A_H}, \sigma_{A_H})$ the corresponding crossed system defined by the convolution invertible total integral $f : H \rightarrow A$. Then there is a bijective correspondence between the second cohomology group $H^2_{\varphi_{Z(A_H)}}(H, Z(A_H))$ and the equivalence classes of crossed systems for H over A having φ_{A_H} as weak H -module algebra structure.*

Proof:

Let $[\tau]$ be in $H^2_{\varphi_{A_H}}(H, Z(A_H))$. Using the properties of the center of A_H , it is not difficult to prove that the morphism $\sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau)$ satisfies conditions (g1) and (g2). As far as (g3), note that

$$\begin{aligned} & (\sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau)) \circ (\Pi_H^L \otimes H) \circ \delta_H \\ &= \mu_{A_H} \circ (\sigma_{A_H} \otimes (i_{Z(A_H)} \circ \tau)) \circ (H \otimes c_{H,H} \otimes H) \circ ((\delta_H \circ \Pi_H^L) \otimes \delta_H) \circ \delta_H \\ &= \mu_{A_H} \circ (\sigma_{A_H} \otimes (i_{Z(A_H)} \circ \tau)) \circ (H \otimes c_{H,H} \otimes H) \circ (((\Pi_H^L \otimes \Pi_H^L) \circ \delta_H) \otimes \delta_H) \circ \delta_H \\ &= \mu_{A_H} \circ ((\sigma_{A_H} \circ (\Pi_H^L \otimes H)) \circ \delta_H) \otimes ((i_{Z(A_H)} \circ \tau) \circ (\Pi_H^L \otimes H) \circ \delta_H) \circ \delta_H \\ &= u_1 \wedge u_1 \\ &= u_1, \end{aligned}$$

where the first equality follows by the definition of the convolution product, in the second one we apply that H is cocommutative and therefore $\delta_H \circ \Pi_H^L = (\Pi_H^L \otimes \Pi_H^L) \circ \delta_H$; the third uses that H is cocommutative; the fourth relies on (61); finally, the last one follows because (A, φ_A) is a weak left H -module algebra. By the equivalence between (60) and (61) we have that $u_1 = (\sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau)) \circ (\eta_H \otimes H)$. In a similar way $u_1 = (\sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau)) \circ (H \otimes \eta_H)$ and then $(\varphi_{A_H}, \sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau))$ is a crossed system for A over H .

Conversely let (φ_{A_H}, γ) be a crossed system for H over A_H . Then the morphism $\sigma_{A_H}^{-1} \wedge \gamma$ factors through the equalizer $i_{Z(A_H)}$. Indeed:

$$\begin{aligned}
& \mu_{A_H} \circ (A_H \otimes \sigma_{A_H}^{-1} \wedge \gamma) \\
&= \mu_{A_H} \circ (A_H \otimes (u_2 \wedge u_2 \wedge \sigma_{A_H}^{-1} \wedge \gamma)) \\
&= \mu_{A_H} \circ (\mu_{A_H} \otimes A_H) \circ (u_2 \wedge u_2 \otimes A_H \otimes \sigma_{A_H}^{-1} \wedge \gamma) \circ (H \otimes c_{A_H, H} \otimes H \otimes H) \circ (c_{A_H, H} \otimes H \otimes H \otimes H) \\
&\quad \circ (A_H \otimes \delta_{H \otimes H}) \\
&= \mu_{A_H} \circ (\mu_{A_H} \otimes H) \circ (\sigma_{A_H}^{-1} \wedge \sigma_{A_H} \wedge u_2 \otimes A_H \otimes \sigma_{A_H}^{-1} \wedge \gamma) \circ (H \otimes c_{A_H, H} \otimes H \otimes H) \circ (c_{A_H, H} \otimes H \otimes H \otimes H) \\
&\quad \circ (A_H \otimes \delta_{H \otimes H}) \\
&= \mu_{A_H} \circ (\mu_{A_H} \otimes H) \circ (\sigma_{A_H}^{-1} \wedge \sigma_{A_H} \otimes (\varphi_{A_H} \circ (H \otimes \psi_{A_H}) \circ ((\delta_H \circ \mu_H) \otimes A_H))) \otimes \sigma_{A_H}^{-1} \wedge \gamma \\
&\quad \circ (\delta_{H \otimes H} \otimes A_H \otimes H \otimes H) \circ (H \otimes c_{A_H, H} \otimes H \otimes H) \circ (c_{A_H, H} \otimes H \otimes H \otimes H) \circ (A_H \otimes \delta_{H \otimes H}) \\
&= \mu_{A_H} \circ (\mu_{A_H} \otimes A_H) \circ ((A_H \otimes (\mu_{A_H} \circ (\sigma_{A_H} \otimes (\varphi_{A_H} \circ (\mu_H \otimes A_H)))) \circ (\delta_{H \otimes H} \otimes A_H)) \otimes A_H \\
&\quad \circ (\sigma_{A_H}^{-1} \otimes H \otimes H \otimes (\varphi_{A_H} \circ (\mu_H \otimes A_H))) \otimes \sigma^{-1} \wedge \gamma \circ (H \otimes H \otimes \delta_{H \otimes H} \otimes A_H \otimes H \otimes H) \\
&\quad \circ (\delta_{H \otimes H} \otimes A_H \otimes H \otimes H) \circ (H \otimes c_{A_H, H} \otimes H \otimes H) \circ (c_{A_H, H} \otimes H \otimes H \otimes H) \circ (A_H \otimes \delta_{H \otimes H}) \\
&= \mu_{A_H} \circ (\mu_{A_H} \otimes A_H) \circ ((A_H \otimes (\mu_{A_H} \circ ((\varphi_{A_H} \circ (H \otimes \varphi_{A_H}))) \otimes A_H) \circ (H \otimes H \otimes c_{A_H, A_H})) \\
&\quad \circ (H \otimes H \otimes \sigma_{A_H} \otimes A_H) \circ (\delta_{H \otimes H} \otimes A_H)) \otimes A_H \circ (\sigma_{A_H}^{-1} \otimes H \otimes H \otimes (\varphi_{A_H} \circ (\mu_H \otimes A_H))) \otimes \sigma_{A_H}^{-1} \wedge \gamma \\
&\quad \circ (H \otimes H \otimes \delta_{H \otimes H} \otimes A_H \otimes H \otimes H) \circ (\delta_{H \otimes H} \otimes A_H \otimes H \otimes H) \circ (H \otimes c_{A_H, H} \otimes H \otimes H) \\
&\quad \circ (c_{A_H, H} \otimes H \otimes H \otimes H) \circ (A_H \otimes \delta_{H \otimes H}) \\
&= \mu_{A_H} \circ (\mu_{A_H} \otimes A_H) \circ (A_H \otimes (\varphi_{A_H} \circ (H \otimes \varphi_{A_H}))) \otimes A_H \\
&\quad \circ (\sigma_{A_H}^{-1} \otimes H \otimes H \otimes (\varphi_{A_H} \circ (\mu_H \otimes A_H))) \otimes \sigma_{A_H} \wedge \sigma_{A_H}^{-1} \wedge \gamma \circ (H \otimes H \otimes \delta_{H \otimes H} \otimes A_H \otimes H \otimes H) \\
&\quad \circ (\delta_{H \otimes H} \otimes A_H \otimes H \otimes H) \circ (H \otimes c_{A_H, H} \otimes H \otimes H) \circ (c_{A_H, H} \otimes H \otimes H \otimes H) \circ (A_H \otimes \delta_{H \otimes H}) \\
&= \mu_{A_H} \circ (A_H \otimes (\mu_{A_H} \circ ((\varphi_{A_H} \circ (H \otimes \varphi_{A_H}))) \otimes A_H) \circ (H \otimes H \otimes c_{A_H, A_H}) \circ (H \otimes H \otimes \gamma \otimes A_H) \\
&\quad \circ (\delta_{H \otimes H} \otimes A_H))) \circ (A_H \otimes H \otimes H \otimes (\varphi_{A_H} \circ (\mu_H \otimes A_H))) \circ (\sigma_{A_H}^{-1} \otimes \delta_{H \otimes H} \otimes A_H) \\
&\quad \circ (H \otimes H \otimes H \otimes c_{A_H, H}) \circ (H \otimes H \otimes c_{A_H, H} \otimes H) \circ (H \otimes c_{A_H, H} \otimes H \otimes H) \circ (c_{A_H, H} \otimes H \otimes H \otimes H) \\
&\quad \circ (A_H \otimes \delta_{H \otimes H}) \\
&= \mu_{A_H} \circ (A_H \otimes (\mu_{A_H} \circ (\gamma \otimes (\varphi_{A_H} \circ (\mu_H \otimes A_H))))) \circ (\delta_{H \otimes H} \otimes A_H) \circ (A_H \otimes H \otimes H \otimes (\varphi_{A_H} \circ (\mu_H \otimes A_H)))
\end{aligned}$$

$$\begin{aligned}
& \circ (\sigma_{A_H}^{-1} \otimes \delta_{H \otimes H} \otimes A_H) \circ (H \otimes H \otimes H \otimes c_{A_H, H}) \circ (H \otimes H \otimes c_{A_H, H} \otimes H) \circ (H \otimes c_{A_H, H} \otimes H \otimes H) \\
& \circ (c_{A_H, H} \otimes H \otimes H \otimes H) \circ (A_H \otimes \delta_{H \otimes H}) \\
& = \mu_{A_H} \circ (\sigma_{A_H}^{-1} \wedge \gamma \otimes (\varphi_{A_H} \circ (H \otimes \varphi_{A_H}) \circ (\delta_H \circ \mu_H) \otimes A_H)) \otimes (\delta_{H \otimes H} \otimes A_H) \circ (H \otimes c_{A_H, H}) \circ (c_{A_H, H} \otimes H) \\
& = \mu_{A_H} \circ (\sigma_{A_H}^{-1} \wedge \gamma \wedge u_2 \otimes A_H) \circ (H \otimes c_{A_H, H}) \circ (c_{A_H, H} \otimes H) \\
& = \mu_{A_H} \circ (\sigma_{A_H}^{-1} \wedge \gamma \otimes A_H) \circ (H \otimes c_{A_H, H}) \circ (c_{A_H, H} \otimes H)
\end{aligned}$$

In the above computations, the first and the third equalities follow because $\sigma_{A_H}^{-1}$ is in $\text{Reg}_{\varphi_{A_H}}(H, A_H)$; the second one because u_2 factors through the center of A_H ; in the fourth and the eleventh ones we use (133); in the fifth and the tenth equalities we apply that H is a weak Hopf algebra; the sixth and ninth rely on (g1) for σ_{A_H} and γ , respectively; the seventh one follows by cocommutativity; the eighth uses that H is cocommutative and σ_{A_H} is a morphism in $\text{Reg}_{\varphi_{A_H}}(H, A_H)$; finally, the twelfth equality follows by (f1) and (f2) for γ .

As a consequence, using that H is cocommutative, $\sigma_{A_H}^{-1} \wedge \gamma \wedge \sigma_{A_H} = \sigma_{A_H} \wedge \sigma_{A_H}^{-1} \wedge \gamma = \gamma$, and therefore $\sigma_{A_H}^{-1} \wedge \gamma = \gamma \wedge \sigma_{A_H}^{-1}$.

The proof for the condition (g2) follows a similar pattern to the one developed in [17] and will be omitted. As far as (g3) the proof follows in a similar way to the one giving for $\sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau)$ using Proposition 3.6.

Finally, we have to show that the correspondence is well defined. Let $[\tau]$ and $[\tau']$ be in $H^2(H, Z(A_H))$ such that the crossed systems $(\varphi_{A_H}, \sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau))$ and $(\varphi_{A_H}, \sigma \wedge (i_{Z(A_H)} \circ \tau'))$ are equivalent. Let h be the morphism in $\text{Reg}_{\varphi_{A_H}}(H, A_H)$ satisfying conditions (74) and (73). Then h factors through the center of A_H . Indeed:

$$\begin{aligned}
& \mu_{A_H} \circ (h \otimes A_H) \\
& = \mu_{A_H} \circ ((h \wedge u_1 \wedge u_1) \otimes A_H) \\
& = \mu_{A_H} \circ ((h \wedge u_1) \otimes (\mu_{A_H} \circ c_{A_H, A_H})) \circ (H \otimes u_1 \otimes A_H) \circ (\delta_H \otimes A_H) \\
& = \mu_{A_H} \circ (\mu_{A_H} \otimes u_1) \circ (h \otimes (\mu_{A_H} \circ ((\varphi_{A_H} \circ (H \otimes \eta_{A_H})) \otimes A_H))) \otimes H) \circ (\delta_H \otimes c_{H, A_H}) \circ (\delta_H \otimes A_H) \\
& = \mu_{A_H} \circ (\mu_{A_H} \otimes u_1) \circ (h \otimes (\varphi_{A_H} \circ (H \otimes \varphi_{A_H}) \circ (\delta_H \otimes A_H))) \otimes H) \circ (\delta_H \otimes c_{H, A_H}) \circ (\delta_H \otimes A_H) \\
& = \mu_{A_H} \circ ((\mu_{A_H} \circ (\mu_{A_H} \otimes A_H) \circ (h \otimes \varphi_{A_H} \otimes h^{-1}) \circ (\delta_H \otimes c_{H, A_H}) \circ (\delta_H \otimes A_H))) \otimes h) \\
& \quad \circ (H \otimes \varphi_{A_H} \otimes H) \circ (\delta_H \otimes c_{H, A_H}) \circ (\delta_H \otimes A_H) \\
& = \mu_{A_H} \circ (\varphi_{A_H} \otimes h) \circ (H \otimes \varphi_{A_H} \otimes H) \circ (\delta_H \otimes c_{H, A_H}) \circ (\delta_H \otimes A_H) \\
& = \mu_{A_H} \circ ((\mu_{A_H} \circ (u_1 \otimes A_H)) \otimes h) \circ (H \otimes c_{H, A_H}) \circ (\delta_H \otimes A_H) \\
& = \mu_{A_H} \circ (A_H \otimes (u_1 \wedge h)) \circ c_{H, A_H} \\
& = \mu_{A_H} \circ (A_H \otimes h) \circ c_{H, A_H}.
\end{aligned}$$

In the foregoing computations, the first and the last equalities follow because h is in $\text{Reg}_{\varphi_{A_H}}(H, A_H)$; the second, third and eight ones use the definition of u_1 and that this morphism factors through the center of A_H ; the fourth and seventh equalities rely on (134); the fifth is a consequence of cocommutativity of H ; finally, the sixth one follows by (73).

Using that h factors through the center of A_H and by (74) it is not difficult to see that τ and τ' are cohomologous.

Conversely, if τ and τ' are cohomologous, by the properties of the center of A_H we get that the corresponding crossed systems $(\varphi_{A_H}, \sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau))$ and $(\varphi_{A_H}, \sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau'))$ are equivalent and we conclude the proof. \square

ACKNOWLEDGEMENTS

The authors were supported by Ministerio de Ciencia e Innovación (Project: MTM2010-15634) and by FEDER.

REFERENCES

- [1] A. Agore, G. Militaru, Extending structures II: The quantum version, arXiv:1011.2174v3 (2011)
- [2] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez, A.B. Rodríguez Raposo, Weak C -cleft extensions, weak entwining structures and weak Hopf algebras. *J. of Algebra* 284 (2005) 679-704.
- [3] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez, A.B. Rodríguez Raposo, Weak C -cleft extensions and weak Galois extensions, *J. of Algebra* 299 (2006) 276-293.
- [4] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez, A.B. Rodríguez Raposo, Crossed products in weak contexts, *Appl. Cat. Structures* 18 (2010), 231-258.
- [5] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez, Weak braided Hopf algebras. *Indiana Univ. Math. J.* 57 (2008) 2423-2458.
- [6] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez, Cohomology of algebras over weak Hopf algebras, arXiv:1206.3850 (2012)
- [7] R.J. Blattner, M. Cohen, S. Montgomery, Crossed products and inner actions of Hopf algebras. *Trans. Amer. Math. Soc.* 298 (1986) 671-711.
- [8] R. Blattner, S. Montgomery, Crossed products and Galois extensions of Hopf algebras, *Pacific J. of Math.* 137 (1989) 37-54.
- [9] G. Böhm, F. Nill, K. Szlachányi, Weak Hopf algebras, I: Integral theory and C^* -structures. *J. of Algebra* 221 (1999) 385-438.
- [10] G. Böhm, The weak theory of monads, *Advances in Math.* 225 (2010), 1-32.
- [11] T. Brzeziński, Crossed products by a coalgebra, *Comm. in Algebra* 25 (1997) 3551-3575.
- [12] T. Brzeziński, On modules associated to coalgebra Galois extensions, *J. of Algebra*, 215 (1999), 290-317.
- [13] D. Bulacu, The weak braided Hopf algebra structure of some Cayley-Dickson algebras, *J. of Algebra*, 322 (2009), 2404-2427.
- [14] D. Bulacu, A Clifford algebra is a weak Hopf algebra in a suitable symmetric monoidal category, *J. of Algebra*, 332 (2011), 244-284.
- [15] S. Caenepeel, E. De Groot, Modules over weak entwining structures. *New Trends in Hopf Algebra Theory*, N. Andruskiewitsch, W.R. Ferrer Santos and H-J. Schneider (eds.), *Contemp. Math.* 267 (2000) 4701-4735.
- [16] Y. Doi, Cleft comodule algebras and Hopf modules, *Comm. in Algebra* 12 (1984) 1115-1169.
- [17] Y. Doi, Equivalent crossed products for a Hopf algebra. *Comm. in Algebra* 17 (1989) 3053-3085.
- [18] Y. Doi, T. Takeuchi, Cleft comodule algebras for a bialgebra. *Comm. in Algebra* 14 (1986), 801-817.
- [19] J.M. Fernández Vilaboa, R. González Rodríguez, A.B. Rodríguez Raposo, Preunits and weak crossed products. *J. of Pure Appl. Algebra* 213 (2009) 2244-2261.
- [20] J.M. Fernández Vilaboa, R. González Rodríguez, R., A.B. Rodríguez Raposo, Partial and unified crossed products are weak crossed products. *Contemporary Mathematics, Proceedings of the Conference: Hopf Algebras and Tensor Categories* (2012) (in press, available at arXiv:1110.6724)
- [21] S. Lack, R. Street, The formal theory of monads II, *J. of Pure Appl. Algebra* 175 (2002) 243-265.
- [22] S. Montgomery, Hopf algebras and their actions on rings, *CBMS* 82 (1992)
- [23] M. Muniz S. Alves, E. Batista, M. Dokuchaev, A. Paques, Twisted partial actions of Hopf algebras, arXiv:1111.1281 (2011).
- [24] D. Nikshych, L. Vainerman, Finite Quantum Groupoids and their applications, *New Directions in Hopf Algebras*, MSRI Publications 43 (2002) 211-262.
- [25] A.B. Rodríguez Raposo, Crossed products for weak Hopf algebras, *Comm. in Algebra* 37 (2009), 2274-2289.
- [26] M. Sweedler, Cohomology of algebras over Hopf algebras. *Trans. of the American Math. Soc.* 133 (1968) 205-239.